E X P A N S I V E   F L O W S

by

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** TO MY PARENTS **

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A B S T R A C T

In this paper we attempt to develop a theory of expansive continuous flows on compact metric spaces analogous to the theory for discrete expansive flows.

In Section 1, we develop the concept of expansiveness for continuous flows, and consider some examples of this concept.

In Section 2, we use the ideas of local cross-sections and flow boxes for a continuous flow to derive a special type of open cover for a space admitting a fixed point free flow. We define what it means for such an open cover to be a generator and show that a flow is expansive if and only if it admits a generator.

In Section 3, we prove some elementary theorems about expansive flows, including the result that, for am expansive flow, there is an exponential growth rate on the number of periodic orbits.

In Section 4, we consider asymptotic properties of expansive flows, and show in particular that the nonisolated closed orbits of an expansive flow are topologically sources, saddles, or sinks.

In Section 5, we prove that every expansive fixed point free flow is a factor of the flow obtained by suspending the shift map on a subspace of the space of sequences modelled on a finite set.

In Section 6, we consider the question of the existence of expansive flows on compact 2-manifolds, and prove that there are no such.

In the Appendix we give a proof (due essentially to D.V. Anosov) that Anosov flows are expansive.

C O N T E N T S

§1. Expansive Homeomorphisms and Continuous Flows

§1.1. Continuous Flows.

Definition. Let (X,d) be a metric space and $\underline{R}$ denote the set of real numbers. A continuous flow on X is a continuous map $\phi : \underline{R} \times X \to X$ such that

(1) For all x in X, $\phi(0,x) = x$

(2) For all x in X, and for all s and t in $\underline{R}$
$\phi(s + t,x) = \phi(s, \phi(t,x))$.

Definition. If ϕ is a continous flow on X, the ϕ-orbit of a point x in X = $\bigcup_{t \text{ in } \underline{R}} \{ \phi(t,x) \}$

Notation. From now on X will denote a metric space with understood metric d. (X,ϕ) will denote a metric space X together with a continuous flow ϕ on X. We will denote the ϕ-orbit of x in X by $O_\phi(x)$ or simply by $O(x)$ if there is no danger of confusion with another flow ψ.

$\underline{R}$ will denote the set of real numbers. For a fixed t in $\underline{R}$ we will write ϕ_t for the map $\phi|\{t\} \times X \to X$. i.e. For all x in X, $\phi_t\, x = \phi(t,x)$. We note that, for each t in $\underline{R}$, ϕ_t is a homeomorphism from X to itself.

§1.2. Expansive Homeomorphisms.

Definition. 1.21. A homeomorphism h of a metric space X is said to be expansive if the following statement holds. There exists $c > 0$ in $\underline{R}$, (the so-called expansive constant for (X,h), such that for all x and y in X with $x \neq y$, there exists an integer n with $d(h^n x, h^n y) > c$.

Important examples of expansive homeomorphisms are the so-called Anosov diffeomorphisms and the shift map on the (compact) metric space of bi-infinite sequences modelled on a finite set.

§1.3 Expansive Flows.

We would like to develop a concept of expansiveness for flows that would include Anosov flows and suspensions of expansive homeomorphisms as examples of expansive flows.

The immediate analogy with Definition 1.21 is comparatively useless: an infinite compact space X does not admit a continuous flow ϕ which is expansive in the following sense:

There exists $c > 0$ in $\underline{R}$ such that for all x and y in X with $x \neq y$, there exists t in R with $d(\phi_t x, \phi_t y) > c$.

For suppose such a constant c existed. If X is compact, for all p in $\underline{R}$ sufficiently close to 0, $d(\phi_p; id_x) < c$.

i.e. for all x in X, $d(\phi_p x, x) < c$.

Thus for all p in $\underline{R}$ with $|p|$ sufficiently small ,

$d(\phi_p(\phi_t x), \phi_t x) < c$ for all x in X, and for all t in $\underline{R}$.

i.e. $d(\phi_t \phi_p x, \phi_t x) < c$ for all t in $\underline{R}$

$\therefore \phi_p x = x$.

Hence for all x in X, and for all $|p|$ sufficiently small,

$$\phi_p x = x .$$

Therefore ϕ is the trivial flow (i.e. every point is fixed under ϕ) on a finite set of points.

Remark. More generally, if we had a continuous abelian group G acting on X, the same argument would apply to show that if X were compact infinite, it would not support an expansive G-action.

This result is stated in [Keynes and Robertson 1] and also occurs in [Gottschalk and Hedlund 1.]

It is clear that if we wish to extend the concept of expansiveness to continuous flows, we need to work "modulo the orbits" in some sense. We are led therefore, to try the following definition: A continuous flow ϕ on X is expansive if there exists $c > 0$ in $\underline{R}$ such that for all x and y in X with $O(x) \neq O(y)$, there exists t in $\underline{R}$ with $d(\phi_t x, \phi_t y) > c$.

However, consider the following example, due to P. Walters.

Example 1.31.

Let X be the unit square in $\underline{R}^2$ with top and bottom sides identified $((x,0) \sim (x,1))$, and ϕ the flow on X given by

$$\phi_t(x,y) = (x, (y + t) \bmod 1),$$

and Y the trapezium in $\underline{R}^2$ bounded by $y = 0$, $x = 0$, $x = 1$, $y = x + 1$, again with top and bottom sides identified $((x,0) \sim (x, x + 1))$, and ψ

the flow on Y given by

$$\psi_t(x,y) = (x,(y + t) \bmod (x + 1)).$$

The map $h : X \to Y$ defined by

$$h(x,y) = (x, y\cdot(x + 1))$$

is a homeomorphism mapping ϕ-orbits onto ψ-orbits, i.e. (X,ϕ) and (Y,ψ) are topologically conjugate.

However, (Y,ψ) is expansive in the above sense, whereas (X,ϕ) is not.

Therefore the definition above is not an invariant with respect to topological conjugacy.

The preceding discussion motivates us to define an expansive flow with

<u>Definition 1.32</u>. A continuous flow ϕ on a metric space X is <u>expansive</u> if

For all $\varepsilon > 0$, there exists $\delta > 0$, such that whenever x and y are in X and ~~there exists~~ for any a homeomorphism $\alpha : \underline{R} \to \underline{R}$ with $\alpha(0) = 0$ and $\alpha(t) > 0$ for $t > 0$, then

$[d(\phi_t x, \phi_{\alpha(t)} y) < \delta$ for all t in $\underline{R}]$ implies that $y = \phi_p x$, where $|p| < \varepsilon$.

<u>Notation</u>. We shall say that α is in $\hom(\underline{R})$ if $\alpha : \underline{R} \to \underline{R}$ is a homeomorphism from $\underline{R}$ to $\underline{R}$ with $\alpha(0) = 0$ and $\alpha(t) > 0$ for $t > 0$.

Remark 1.33. An equivalent statement to Definition 1.32 is to say that a continuous flow ϕ on a metric space X is expansive if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever x and y are in X and α and β are in hom($\underline{R}$), $[d(\phi_{\alpha(t)}x, \phi_{\beta(t)}y) < \delta$ for all t in $\underline{R}]$ implies that $y = \phi_p x$, where $|p| < \varepsilon$.

Remark 1.34. Bowen and Walters have defined expansiveness for continuous flows in a slightly different way as follows:

(X,ϕ) is flow expansive if

For all $\varepsilon > 0$, there exists $\delta > 0$ such that $y = \phi_p x$ for some p with $|p| < \varepsilon$, whenever $s : \underline{R} \to \underline{R}$ is continuous with $s(0) = 0$ and $d(\phi_{t+s(t)}y, \phi_t x) < \delta$, for all t in $\underline{R}$. (see [Bowen & Walters 1]).

Theorem 1.35. If ϕ is an expansive flow on a metric space X, then the fixed points of ϕ are isolated, and if X is compact, there are a finite number of fixed points.

Proof. The second statement is clear, and to prove the first, we suppose the contrary holds. Let x in X be a fixed point of ϕ, and $\{x_n\}$ a sequence of points in X converging to x.

Consider a fixed $\varepsilon > 0$. We have to show that for all $\delta > 0$, we can find y and w in X and α in hom($\underline{R}$) with $y \neq \phi_p w$ for any p with $|p| < \varepsilon$, and

$$d(\phi_t w, \phi_{\alpha(t)}y) < \delta \quad \text{for all t in } \underline{R}.$$

Suppose $\delta > 0$ is given. By continuity of the flow, there exists an integer N such that, for all $n > N$,

$d(\phi_t x_n, \phi_s x_n) < \delta$ for all s and t in $\underline{R}$

with $|s|, |t| \leq 2\varepsilon$.

We construct α in hom($\underline{R}$) as follows :

$\alpha(t) = 2t$ for $0 < t \leq \varepsilon$

$\alpha(t) = \frac{1}{2}t$ for $-2\varepsilon < t < 0$

$\alpha(t) = t + \varepsilon$ for t otherwise

Then $d(\phi_t(\phi_\varepsilon x_n), \phi_{\alpha(t)} x_n) < \delta$ for all t in $\underline{R}$ and for all x_n with $n > N$.

We have two cases.

1. We can find x_n with $n > N$ and x_n not periodic of period $\leq \varepsilon$, in which case we are done.

2. Each x_n, $n > N$, is periodic of period $\leq \varepsilon$.
But $x_n \to x$ and x is fixed. Therefore, given $\delta > 0$, there exists an integer M, such that for all $n > M$, $d(\phi_t x_n, \phi_t x) < \delta$ for all t in $\underline{R}$, and (X,ϕ) is thus not expansive.

1.4 Examples of Expansive Flows.

1.41 Flows on S^1. Any fixed point free flow on the circle is expansive. For there is only one orbit, and given $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x,y) < \delta$ implies that $y = \phi_p x$, where $|p| < \varepsilon$.

Neither flow in Example 1.31 is expansive.

1.43. Suspensions of Expansive Homeomorphisms.

Recall that if $h : X \to X$ is a homeomorphism, the suspended flow, (or suspension of h), σ^h is defined on the identification space $Y = (X\times I)/_\sim$ where $(x,1) \sim (hx, 0)$, and σ_t^h is defined by

$\sigma_t^h(x,s) = (h^n x, r)$ where n is the quotient and r the remainder on dividing $s + t$ by 1.

Theorem 1.431. $h : X \to X$ is expansive if and only if the suspended flow σ^h on Y is expansive.

We defer the proof of this theorem until §3.2.

1.44. Anosov Flows.

We recall the definition of an Anosov flow. A differentiable flow ϕ on a compact (Riemannian) manifold M is said to be an Anosov Flow if the tangent bundle TM of M splits continuously into three subbundles invariant with respect to $d\phi_t$, say $TM = X \oplus Y \oplus Z$ where Z is the 1 dimensional bundle along the flow, and if the tangent plane at w in M is $T_wM = X_w \oplus Y_w \oplus Z_w$, than for ξ in X_w, ζ in Y_w, we have

(i) $|d\phi_t(\xi)| \leq a\ |\xi|e^{-ct}$, $t \geq 0$

$|d\phi_t(\xi)| \geq b\ |\xi|e^{-ct}$, $t \leq 0$

and (ii) $|d\phi_t(\zeta)| \leq a\ |\zeta|e^{ct}$, $t \leq 0$

$|d\phi_t(\zeta)| \geq b\ |\zeta|e^{ct}$, $t \geq 0$.

The constants a,b,c are positive and are the same for all w, ξ and ζ.

<u>Theorem 1.441</u>. Anosov flows are expansive.

<u>Proof</u>. We give a proof of this theorem in the Appendix, the details of which are extracted from D.V. Anosov's monograph on geodesic flows on manifolds with negative curvature (see [Anosov 1]).

~~Corollary 1.442~~. A related result with a similar proof is
Theorem 1.442

If ϕ is one of Smale's Axiom A' flows, (see [Smale 1]), then $\phi|\Omega(\phi)$ is expansive, where $\Omega(\phi)$ is the non-wandering set of ϕ.

§2. Local Sections and Generators

§2.0 Introduction. In this section all spaces will be compact. We develop the notion of a generator or separating open cover for a continuous fixed point free flow on a compact space which is analogous to the concept of a generator for a discrete flow. We recall that if $h : X \to X$ is a homeomorphism, a generator for (X,h) is a finite open cover $\mathcal{U}$ such that if (U_i) is any bi-infinite sequence of members of $\mathcal{U}$, then

$$\bigcap_{i=-\infty}^{\infty} h^{-i}\,\bar{U}_i \text{ is at most one point.}$$

Theorem 2.01. (See, for example, [Bryant & Walters 1]).

A homeomorphism $h : X \to X$ is expansive if and only if (X,h) admits a generator.

2.1. Local Sections and Flow Boxes.

The notion of a local section for a continuous flow is due to Whitney, (see [Whitney 1]), and the idea has been developed in greater detail by Beboutov, amongst others. (See [Nemytskii & Stepan

Definition 2.11. Let (X,ϕ) be a continuous flow. For x in X, a closed set S_x containing x is a local section through x if x has a closed neighbourhood $N(x)$ in X homeomorphic to the product

$$S_x \times [-\mu, \mu], \text{ where } \mu \text{ is in } \underline{R}^+.$$

i.e. If y is in $N(x)$, then there exists a unique time t_y such that

$|t_y| \leq \mu$ and $\phi_{t_y} y$ is in S_x, and the map

$$k : N(x) \to S_x \times [-\mu, \mu]$$

sending $y \to (\phi_{t_y} y, t_y)$

is a homeomorphism.

We shall say $N(x)$ is a flow box neighbourhood of x of length 2μ.

Theorem 2.12 (Beboutov). Let ϕ be a continuous flow on X, and let x in X be a non critical point of the flow. If x is not a periodic point of ϕ, then x admits a flow box neighbourhood of arbitrary length. If x is periodic of period 4μ say, then x admits a flow box neighbourhood of length 2τ, for any positive $\tau < \mu$.

Remark. If $\overline{B(x,\delta)}$ denotes the closed ball neighbourhood of x of radius δ, and S_x is a local section through x, then $\overline{B(x,\delta)} \cap S_x$ is also a local section through x. Thus we may choose local sections of as small a diameter as we please.

Notation. We shall write $F(S,\mu)$ for a set in X which is a flow box neighbourhood of some point in X. $F(S,\mu)$ has an associated local section S, and is of length 2μ.

i.e. $F(S,\mu) = S \times [-\mu,\mu]$.

π^1 will denote the projection map from F to S.

π^2 will denote the projection map from F to $[-\mu,\mu]$.

If $F_x = S_x \times [-\mu,\mu]$ is a flow box neighbourhood of x, we denote the interior of F_x by $\mathring{F}_x$. $\mathring{F}_x$ will be an open neighbourhood of x.

Lemma 2.13. If ϕ is a fixed point free flow on a compact space X, then there is a minimum period for a periodic orbit.

Proof. Suppose not. Then there exists a sequence of points x_i in X and times t_i with $t_i \to 0$ and x_i periodic of period t_i. We may suppose $x_i \to x$ in X.

Now x is not fixed, hence by theorem 2.12, x has a flow box neighbourhood $F_x = S_x \times [-\mu, \mu]$.

There exists $N_1 > 0$, such that for $i > N_1$, x_i is in F_x.

There exists $N_2 > 0$, such that for $i > N_2$, $0 < t_i < \mu$.

Therefore, for $i > N_1$ and N_2, x_i is in F_x and

$$\phi_{t_{x_i}}(x_i) \text{ is in } S_x \text{ with } |t_{x_i}| \leq \mu.$$

Also $\phi_{t_i + t_{x_i}}(x_i)$ is in S_x, since x_i is periodic of period t_i. This contradicts the definition of S_x, since $\phi_t(\phi_{t_i} x_i)$ cannot belong in S_x for $0 < |t| < 2\mu$.

Corollary 2.14. If ϕ is a fixed point free flow on a compact space X, then there exists $\tau > 0$ such that each point x in X has a flow box neighbourhood of length 2τ. Proof. Take 4τ equal to the minimum period of a periodic orbit and apply Theorem 2.12.

Definition 2.15. Let ϕ be a fixed point free flow on a compact space X. A *μ flow box cover* $\mathcal{F}$ for (X,ϕ) is a finite open cover $\mathcal{F} = \{\overset{\circ}{F}_1, \ldots, \overset{\circ}{F}_p\}$ of X such that

(1) For each i, F_i is a flow box of length 2μ i.e. $F_i = S_i \times [-\mu,\mu]$, and

(2) $F_i' = S_i \times [-3\mu,3\mu]$ is also a flow box.

Theorem 2.16. Let ϕ be a fixed point free flow on a compact space X. Then there exists $\mu_0 > 0$, such that if $0 < \mu \leq \mu_0$, (X,ϕ) admits a μ flow box cover.

Proof. By compactness, Theorem 2.12 and Corollary 2.14.

§2.2. Generators and Expansiveness.

In this section we define a generator for a continuous flow as a flow box cover which satisfies a further condition, and show that a fixed point free flow or a compact space X is expansive if and only if (X,ϕ) admits a generator.

Definition 2.21. Let ϕ be a fixed point free flow on a compact space X and let $= \{\overset{\circ}{F}_1, \ldots, \overset{\circ}{F}_p\}$ be a μ flow box cover for (X,ϕ), with each F_i having associated local section S_i. If *$x \in X$, we say x admits a sequence (u_i)* if there exists a sequence of times (t_i) with $0 \leq |t_o| \leq \mu$ and $\mu/4 \leq t_i - t_{i-1} \leq 3\mu$, such that, for all integers i, $\phi_{t_i} x$ is in S_{u_i}

Remark. Intuitively, x admits a sequence (u_i), if $O(x)$ runs through the sequence of flow boxes (F_{u_i}).

Definition 2.22. Let ϕ be a fixed point free flow on a compact space X. A μ flow box cover $\mathcal{F} = \{\overset{o}{F}_1, \ldots, \overset{o}{F}_p\}$ for (X,ϕ) is a generator, or more strictly a μ-generator, if whenever x and y in X admit a common sequence, then $y = \phi_p x$, where $|p| \leq 2\mu$.

Theorem 2.23. A fixed point free flow on a compact space X is expansive if and only if (X,ϕ) admits a generator.

Proof. Suppose (X,ϕ) admits a μ-generator $\mathcal{F} = \{\overset{o}{F}_1, \ldots, \overset{o}{F}_p\}$. The idea is to show that if δ is chosen sufficiently small, then if x and y are in X and there exists α in hom($\underline{R}$)

with $d(\phi_t x, \phi_{\alpha(t)} y) < \delta$ for all t in $\underline{R}$,

then x and y admit a common sequence of sections.

Let $\delta_1 > 0$ be a Lebesgue number for $\mathcal{F}$. Let π_i^1 and π_i^2 denote the projections onto the first and second factors respectively of

$$F_i' = S_i \times [-3\mu, 3\mu].$$

By the local product structure for each F_i', there exists $\delta_2 > 0$ such that if x and y are in F_i' and $d(x,y) < \delta_2$, then $|\pi_i^2(x) - \pi_i^2(y)| < \mu/4$. For the same reason, there exists $\delta_3 > 0$, such that if x and y are in F_i', and $d(x,y) < \delta_3$, then $d(\pi_i^1 x, \pi_i^1 y) < \delta_2$.

Now suppose we have x and y in X, and α in hom($\underline{R}$) with $d(\phi_t x, \phi_{\alpha(t)} y) < \min\{\delta_1, \delta_3\}$.

Then x and y are in $\overset{o}{F}_o$ say.

Let $x_o = \pi_o^1(x)$ in S_o.

$y_o = \pi_o^1(y)$ in S_o.

Now $\phi_{3\mu/2}(x_o) = \phi_{s_1} x$ say, and $\phi_{\alpha(s_1)} y$, lie in a common (open) flow box $\overset{o}{F}_1$ where $F_1 \neq F_o$.

Therefore there exists t_1 with $\mu/2 < t_1 < 5\mu/2$ and $\phi_{t_1} x_o$ lying in S_1.

Also there exists r_1 with $\phi_{r_1} y_o$ lying in S_1, and by choice of $\delta < \delta_3$, $\mu/4 < r_1 < 11\mu/4$.

Writing $x_1 = \phi_{t_1} x_o$ and $y_1 = \phi_{r_1} y_o$, we can repeat this process by considering $\phi_{3\mu/2} x_1$, etc.

Hence we can build up a common sequence of sections for x and y.

Finally, we note that since (X,ϕ) is fixed point free, if $\varepsilon > 0$ is given, there exists $\delta_4 > 0$ such that, if x and y are in X and $y = \phi_p x$, where $|p| < 2\mu$, then $d(x,y) < \delta_4$ implies that $|p| < \varepsilon$.

Hence choosing $\delta = \min\{\delta_1, \delta_3, \delta_4\}$, if x and y are in X and α is in hom($\underline{R}$) with $d(\phi_t x, \phi_{\alpha(t)} y) < \delta$ for all t in $\underline{R}$, then $y = \phi_p x$, where $|p| < \varepsilon$.

Conversely, we have to show that if (X,ϕ) is expansive and fixed point free, then (X,ϕ) has a generator.

There exists $\mu_o > 0$ such that for $0 < \mu \leq \mu_o$, (X,ϕ) admits a μ-flow box cover. We have to choose a flow box cover that will be a generator. We do this by choosing local sections of sufficiently small diameter.

Take $\varepsilon = 2\mu_o$. There exists $\delta > 0$, such that if there exist x,y in X and α in hom($\underline{R}$) with

$d(\phi_t x, \phi_{\alpha(t)} y) < \delta$ for all t in $\underline{R}$, then

$y = \phi_p x$, where $|p| < 2\mu_o$.

There exist θ and $\delta_1 > 0$ such that if $d(x,y) < \delta_1$ and $|s - t| < \theta$, then

$d(\phi_t x, \phi_s y) < \delta$ for $|s,t| \leq 3\mu_o$, where δ is as above.

There exists $\xi > 0$, such that if $F(S,\tau)$ is a flow box in X of length $\leq 6\mu_o$, then x,y in F and $d(x,y) < \xi$

imply that $|\pi^2(x) - \pi^2(y)| < \theta$.

Choose a μ_o flow box cover $\mathcal{F} = \{\overset{o}{F}_1 \ldots \overset{o}{F}_p\}$ where each flow box F_i has an associated local section S_i of diameter $\leq \min\{\delta_1, \xi\}$.

Suppose x and y admit a common sequence of sections (S_{u_i}).

Consider $x_o = \pi_o^1(x)$ in S_{u_o}

$y_o = \pi_o^1(y)$ in S_{u_o}.

Then $\phi_{s_1} x_o$ is in S_{u_1} for some s_1, $\mu/4 \leq s_1 \leq 3\mu_o$

$\phi_{t_1} y_o$ is in S_{u_1} for some t_1, $\mu/4 \leq t_1 \leq 3\mu_o$.

By construction, $d(x_o, y_o) < \delta$, and therefore

$|s_1 - t_1| < \theta$.

We define $\alpha: [0,s_1] \to [0, t_1]$ by

$$\alpha(t) = t \cdot \frac{t_1}{s_1}$$

Then for t in $[0,s_1]$, $|t - \alpha(t)| = \frac{t}{s_1} |s_1 - t_1| < \theta$.

Thus for t in $[0,s_1]$, $d(\phi_t x_o, \phi_{\alpha(t)} y_o) < \delta$.

Correspondingly, we have $\phi_{s_2} x_o$ in S_{u_2},

$\phi_{t_2} y_o$ in S_{u_2},

with $\mu/_4 \leq s_2 - s_1 \leq 3\mu_o$, $\mu/_4 \leq t_2 - t_1 \leq 3\mu$,

and for t in $[s_1, s_2]$, we define $\alpha(t)$ by

$$\alpha(t) = \frac{(t_2 - t_1)(t - s_1)}{s_2 - s_1} + t_1 ,$$

i.e. α maps $[s_1, s_2]$ linearly onto $[t_1, t_2]$ and as before $d(\phi_t x_o, \phi_{\alpha(t)} y_o) < \delta$.

We continue in this way, inductively building a homeomorphism α in hom($\underline{R}$)

with $d(\phi_t x_o, \phi_{\alpha(t)} y_o) < \delta$ for all t in $\underline{R}$.

$\therefore$ $y_o = \phi_p x_o$, where $|p| < 2\mu$.

But x_o and y_o are in S_{u_o}.

$\therefore$ $y_o = x_o$, and $\mathcal{J}$ is a generator.

<u>Note</u>. In future work, if X is an expansive flow on a compact space X, and we refer to a generator for (X,ϕ), we shall mean a generator for $(X \setminus F_\phi, \phi)$, where F_ϕ is the fixed point set of ϕ.

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§3. Properties of Expansive Flows

§3.0. Introduction.

In this section we develop some of the properties of expansive flows on compact spaces. We show that expansiveness is an invariant with respect to topological conjugacy, and consider a related concept to expansiveness, which we term weak expansiveness. We show that for flows admitting a global section, weak expansiveness is equivalent to expansiveness.

We show that in general, factors of expansive flows are not expansive, unless the projection map is locally a homeomorphism.

Next we look at maps and flows commuting with an expansive flow.

We also consider periodic orbits for an expansive flow and show that there is an exponential bound on the growth rate of the number of periodic orbits.

§3.1. Topological Conjugacy.

Lemma 3.11. Let ϕ be a fixed point free continuous flow on a compa space X, and let ψ on Y be a topologically conjugate flow, i.e., there exists a homeomorphism $h : X \to Y$ mapping ϕ-orbits to ψ-orbits and preserving the sense of the orbits. Then there exists a unique continuous map $\alpha : \underline{R} \times X \to \underline{R}$ such that

(1) For all t in $\underline{R}$, and for all x in X,

$$h\,\phi_t\, x = \psi_{\alpha(t,x)} hx.$$

(2) For a fixed x in X, the map $\alpha_x : \underline{R} \to \underline{R}$ defined by $\alpha_x(t) = \alpha(t,x)$, is in hom($\underline{R}$).

(3) For all x in X, and for all s and t in $\underline{R}$.

$$\alpha(s + t), x) = \alpha(s, \phi_t x) + \alpha(t,x).$$

Remark. If α exists satisfying (1), it must satisfy (3), since ϕ and ψ are flows. We therefore construct a unique α on (a neighbourhood) of zero in $\underline{R}$) $\times$ X, and use (3) to extend the domain of α to all of $\underline{R} \times X$.

Proof. For each x in X, we first define α_x on a neighbourhood of zero. Since (X,ϕ) and (Y,ψ) have no fixed points, each point y in Y has a ψ flow box neighbourhood $F_y = S_y \times [-\mu', \mu']$, for some $\mu' > 0$ (independent of y by compactness). Also, each point x in X, has a ϕ flow box neighbourhood $F_x = S_x \times [-\mu,\mu]$ such that

$h(F_x) \subset F_{hx}$, where again μ is independent of x, by compactness

Now h maps orbits to orbits, so h maps $\{\phi_t x : |t| \leq \mu\}$ homeomorphically onto a (connected) piece of the ψ-orbit through hx contained in F_{hx},

i.e. $h\,\phi_t x = \psi_{t'}\, hx$ for t in $[-\mu, \mu,]$,

where t' in $[-\mu',\mu']$ is unique.

$\therefore$ For x in X, and $|t| \leqslant \mu$, define $\alpha(t,x) = t'$.
Clearly $\alpha_x(0) = 0$ and $\alpha_x | [-\mu,\mu]$ is 1 - 1 and continuous, and therefore is a homeomorphism onto its image.

We now have α defined for all x in X and t in $[-\mu,\mu]$.

We extend the domain of definition of α to all of $\underline{R} \times X$, using (3) inductively.

Suppose α has been defined on $X \times [-n\mu,n\mu]$, where n is a positive integer.

Let $t = n\mu + t'$ where t' is in $[0,\mu]$.

Define $\alpha(t,x)$ by $\alpha(t,x) = \alpha(n\mu,x) + \alpha(t',\phi_{n\mu}\, x)$.
Similarly if $t = -n\mu + t'$, where t' is in $[-\mu, 0]$, we define $\alpha(t,x)$ by

$$\alpha(t,x) = \alpha(-n\mu, x) + \alpha(t', \phi_{-n\mu}\, x).$$

This extends the domain of definition of α to

$$X \times [-(n+1)\mu, (n+1)\mu].$$

Since $\alpha_x | [-\mu,\mu]$ is a homeomorphism onto its image, it follows that its extension $\alpha_x : \underline{R} \to \underline{R}$ is also a homeomorphism and in fact is in hom$(\underline{R})$.

We now prove that α is continuous. Firstly, we note that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t| < \delta$, then $|\alpha(t,x)| < \varepsilon$ for all x in X. (For as in the definition of α_x in a neighbourhood of 0, we could take flow boxes in Y of length 2ε, and define corresponding flow boxes in X of length 2δ with $h\,F_x \subset F_{hx}$).

Hence to show α is continuous at (s,x) in $\underline{R} \times X$, it suffices to show that y sufficiently close to x implies that $\alpha(s,y)$ is arbitrarily close to $\alpha(s,x)$. Furthermore, by the inductive definition of α_x and α_y it suffices to prove the above statement for $|s| \leq \mu$.

Consider x in X. Let S_x be the local section associated with the ϕ flow box neighbourhood F_x of x, S_{hx} the local section (w.r.t ψ) through hx. For y in F_x, hy is in F_{hx}. Let $\phi_\tau y$ be the unique point on the ϕ-orbit of y with $|\tau| \leq \mu$ and $h\,\phi_\tau y$ in S_{hx}. For y close to x, τ is close to 0. By the local product structure of F_{hx}, if y is close to x(and hence $\phi_\tau y$ is close to x), $\alpha(t,x)$ is close to $\alpha(t, \phi_\tau y)$.

$$\text{Now}\quad \alpha(t+\tau, y) = \alpha(t,y) + \alpha(\tau, \phi_t y)\,.$$
$$= \alpha(\tau,y) + \alpha(t, \phi_\tau y)\,.$$

For τ close to 0, $\alpha(\tau,y)$ and $\alpha(\tau,\phi_t y)$ are close to 0, and hence $\alpha(t,y)$ is close to $\alpha(t,\phi_\tau y)$.

Hence if x is chosen sufficiently close to y, τ is sufficiently close to 0, so that $\alpha(t,x)$ and $\alpha(t,y)$ are arbitrarily close.

Remark. Anatole Beck has also proved this lemma in the general case i.e. when (X,ϕ) has fixed points. (see his forthcoming book).

Corollary 3.12. If ϕ is an expansive flow on a compact space X, and (Y,ψ) is topologically conjugate via h to (X,ϕ), then the map α in lemma 3.11 exists.

Proof. The fixed points of ϕ are isolated. Hence away from the fixed point set F, α is defined. i.e. α is defined on $\underline{R} \times (X \setminus F)$, and for p in F, we define $\alpha_p : \underline{R} \to \underline{R}$ as the identity on $\underline{R}$.

Theorem 3.13. Expansiveness for flows on compact spaces is an invariant of topological conjugacy.

Proof. Let (Y,ψ) be topologically conjugate to (X,ϕ) via h. Suppose (Y,ψ) is expansive, and (X,ϕ) is not. We shall show this leads to a contradiction.

Remark. We may restrict our attention to $(X \setminus F_\phi, \phi)$ and $(Y \setminus F_\psi, \psi)$, where F_ϕ is the fixed point set of (X,ϕ) and F_ψ is the fixed point set of (Y, ψ).

As in the proof of lemma 3.11, there exist μ and $\mu' > 0$ such that for all x in X, x has a flow box neighbourhood F_x of length 2μ with $h(F_x) \subset F'_{hx}$ where F'_{hx} is a flow box neighbourhood of hx of length $2\mu'$.

Let α be the map described in lemma 3.11,

i.e. $h\,\phi_t x = \psi_{\alpha(t,x)} hx$ for all x in X,

and for all t in $\underline{R}$.

Now (X,ϕ) is not expansive, so there exists $\varepsilon_o > 0$, such that, for all $\delta > 0$, there exist x_1 and x_2 in X, and γ in hom($\underline{R}$) with $x_2 \neq \phi_p\, x_1$ for $|p| < \varepsilon_o$, and $d(\phi_t x_1, \phi_{\gamma(t)}\, x_2) < \delta$ for all t in $\underline{R}$.

Choose $\varepsilon = \min\{\mu, \varepsilon_o\}$.

If $x' \neq \phi_p x$, for any p with $|p| < \varepsilon$, then $hx' \neq \psi_q\, hx$ for any q with $|q| < \alpha(\varepsilon)$ where $\alpha(\varepsilon) = \min\limits_{x\in X} \{\alpha_x(\varepsilon), |\alpha_x(-\varepsilon)|\}$

There exists $\delta_Y > 0$ such that if y_1 and y_2 are in Y and β is in hom($\underline{R}$),

$d(\psi_t y_1, \psi_{\beta(t)}\, y_2) < \delta_Y$ for all t in $\underline{R}$ implies that $y_2 = \psi_q\, y_1$, where $|q| < \alpha(\varepsilon)$.

There exists $\delta_X > 0$, such that if $d(x_1, x_2) < \delta_X$, then $d(hx_1, hx_2) < \delta_Y$.

Select x_1 and x_2 in X with γ in hom($\underline{R}$), $x_2 \neq \phi_p\, x_1$, $|p| < \varepsilon$, and $d(\phi_t\, x_1, \phi_{\gamma(t)}\, x_2) < \delta_X$ for all t in $\underline{R}$.

Then $d(\psi_{\alpha(t,x_1)}(h\, x_1), \psi_{\alpha(\gamma(t),\, x_2)}(h\, x_2)) < \delta_Y$

for all t in $\underline{R}$.

This induces β in hom($\underline{R}$) such that

$d(\psi_t(h\, x_1), \psi_{\beta(t)}(h\, x_2)) < \delta_Y$ for all t in $\underline{R}$.

Thus $h\, x_2 = \psi_q\, h\, x_1$, where $|q| < \alpha(\varepsilon)$,

which is a contradiction.

3.2. Time Change Flows and Weak Expansiveness.

Definition 3.21. A time change flow of a flow (X,ϕ) is a flow (X,ψ) which is topologically conjugate to (X,ϕ) via the identity.

Remarks. If X is compact and (X,ψ) is a time change flow of (X,ϕ) then we have a continuous function $\alpha : \underline{R} \times X \to \underline{R}$ with $\phi_t x = \psi_{\alpha(t,x)} x$ (see §3.1).

(Equally, of course, we have $\beta : \underline{R} \times X \to \underline{R}$ with $\psi_t x = \phi_{\beta(t,x)} x$.)

Notice that if we have a compact flow (X,ϕ), then a continuous map $\beta : \underline{R} \times X \to \underline{R}$ satisfying

(1) $\beta_x : \underline{R} \to \underline{R}$ is in $\hom(\underline{R})$, (where β_x is the map defined by $\beta_x(t) = \beta(t,x)$), and

(2) For all x in X, and for all s and t in $\underline{R}$,

$$\beta(s + t, x) = \beta(s,x) + \beta(t, \phi_{\beta(s,x)}(x))$$

automatically induces a time change flow (X,ψ) of (X,ϕ).

We simply define $\psi_t x = \phi_{\beta(t,x)} x$.

One way of obtaining a time change flow from a flow (X,ϕ) is via a continuous strictly positive function $\lambda : X \to \underline{R}$ as follows.

Define $\beta(t,x)$ to be the unique solution to the equation

$$t = \int_0^{\beta(t,x)} \frac{ds}{\lambda\circ\phi_s x}$$

where the integral is taken with respect to Lebesgue measure on $\underline{R}$ (see [Humphries 1] for details).

<u>Lemma 3.22</u>. Let (X,ϕ) be a continuous compact flow and (Y,ψ) be topologically conjugate to (X,ϕ) via h. Then there is a time change flow (Y,ξ) of (Y,ψ) such that, for all t in $\underline{R}$, $h\,\phi_t = \xi_t h$.

<u>Proof</u>. Let α be the map described in §3.1.,

i.e. $h\,\phi_t x = \psi_{\alpha(t,x)} hx$.

Define $\beta : \underline{R} \times Y \to \underline{R}$ by $\beta(t,y) = \alpha(t, h^{-1}y)$.

Clearly β is continuous and satisfies (1) above.

$$\begin{aligned}
\text{Also } \beta(s+t,y) &= \alpha(s+t,\ h^{-1}y) \\
&= \alpha(s,h^{-1}y) + \alpha(t,\ \phi_s(h^{-1}y)) \\
&= \alpha(s,\ h^{-1}y) + \alpha(t,\ h^{-1}\psi_{\alpha(s,h^{-1}y)}\ (y)) \\
&= \beta(s,y) \quad + \beta(t,\ \psi_{\beta(s,y)}(y)) \ .
\end{aligned}$$

Hence (2) above is satisfied.

Defining $\xi_t y$ as $\psi_{\beta(t,y)}(y)$, we see that

$$h\,\phi_t = \xi_t\,h.$$

<u>Definition 3.23</u>. A continuous flow ϕ on a compact space X is <u>weakly expansive</u> if, for all $\varepsilon > 0$, there exists $\delta > 0$, such that if x and y are in X and ψ is any time change flow of ϕ,

then $d(\psi_t(x), \psi_t(y)) < \delta$ for all t in $\underline{R}$

implies that $y = \phi_p(x)$, where $|p| < \varepsilon$.

<u>Theorem 3.24</u>. If X is compact, and ϕ is a continuous flow on X, then (X,ϕ) expansive implies that (X,ϕ) is weakly expansive.

<u>Proof</u>. Suppose (X,ϕ) is not weakly expansive.

i.e. there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$,

there exist x and y in X and ψ a time change flow of ϕ with $d(\psi_t(x), \psi_t(y)) < \delta$ for all t in $\underline{R}$ and $y \neq \phi_p x$ for any p with $|p| < \varepsilon_0$.

Now $d(\psi_t(x), \psi_t(y)) < \delta$ for all t in $\underline{R}$ means that there exist β_x and β_y in hom($\underline{R}$)

with $d(\phi_{\beta_x(t)}(x), \phi_{\beta_y(t)}(y)) < \delta$ for all t in $\underline{R}$.

Hence (X,ϕ) is not expansive, since β_x, β_y are in hom($\underline{R}$).

<u>Conjecture</u>. It seems likely that the converse of this theorem is also true i.e. if (X,ϕ) is a compact continuous flow then (X,ϕ) weakly expansive implies that (X,ϕ) is expansive.

<u>Theorem 3.25</u>. Weak expansiveness is an invariant of topological conjugacy.

<u>Proof</u>. Let (Y,ψ) be topologically conjugate to (X,ϕ) via h. Then there is a time change flow ψ' of ψ such that $h\phi_t = \psi'_t h$. Moreover, if ξ is a time change flow of ϕ, the ξ induces a time change flow ζ of ψ such that

$$h\xi_t = \zeta_t h \text{ for all } t \text{ in } \underline{R} \text{ and } x \text{ in } X..$$

(The proof of this is similar to that of Lemma 3.22).

The remainder of the proof is similar to that of Theorem 3.13.

<u>Definition</u>. If ϕ is a continuous flow on X, a <u>global section</u> for (X,ϕ) is a closed set $K \subset X$ such that if ξ is the map from $\underline{R} \times K \to X$ defined by $\xi(t,k) = \phi_t k$, then ξ is a surjective local homeomorphism.

<u>Remarks</u>. If K is a local section for (X,ϕ), then the map ξ is automatically a local homeomorphism, and hence a local section K is a global section if, for every x in X, the orbit of x meets K.

Clearly a global section is a local section.

If a compact continuous flow (X,ϕ) admits a global section K, then the continuous flow ϕ can be studied by looking at the discrete flow (K,g) where, for k in K, $g(k) = \phi_{t_k} k$, where t_k is the time of first return of k to K.

Conversely, if $g : X \to X$ is a homeomorphism, then the suspended flow σ^g described in §1.43 admits $X \times \{0\}$ as a global section, with time of first return of each point in $X \times \{0\}$ equal to 1.

Lemma 3.27. If ϕ is a continuous flow on a compact space X, admitting a global section K, then there is a time change flow ψ of ϕ such that, for all k in K, the ψ-time of first return of k to K is 1.

Proof. For k in K, let the ϕ-time of first return of k to K be t_k. Consider the space $Y = (K \times I)/\sim$ where $\sim$ denotes the identification g say : $(k,1) \sim (\phi_{t_k} k, 0)$.

The map $h : X \to Y$ defined by

$$(k, \phi_s k) \mapsto (k, {}^s/t_k) \quad \text{for } k \text{ in } K,$$
$$0 \leqslant s \leqslant t_k$$

is a conjugacy from (X,ϕ) onto (Y,σ^g).

For points in h(K), the time of first return under σ_g is 1, and hence by Lemma 3.22, the result follows.

Theorem 3.28. Let (X,ϕ) be a compact continuous flow admitting a global section K. Then (X,ϕ) is expansive if and only if (X,ϕ) is weakly expansive. Moreover, (X,ϕ) is expansive if and only if the induced discrete flow on K is expansive.

Proof. Using Theorem 3.13, Theorem 3.25, and Lemma 3.27, it suffices to prove this result for the case, where, for all k in K, the ϕ-time of first return of k to K is 1.

Let h denote the induced homeomorphism on K. Suppose h is not expansive. If $\delta > 0$ is given, there exists $\delta_1 > 0$ such that $d(x,y) < \delta_1$ implies that $d(\phi_t x, \phi_t y) < \delta$ for all t, $|t| \leq 1$. Choose x and y in K, (therefore $y \neq \phi_p x$, $|p| < 1$), with $d(h^n x, h^n y) < \delta_1$ for all integers n. Thus

$d(\phi_t x, \phi_t y) < \delta$ for all t in $\underline{R}$. Hence (X,ϕ) is not weakly expansive and therefore not expansive.

Conversely, suppose (K,h) is expansive with expansive constant δ_o. K is a local section and $F = K \times [-\frac{1}{2}. \frac{1}{2}]$ a flow box. Therefore there exist δ_1 and $\delta_2 > 0$ such that if x is in K and $d(x,y) < \delta_1$, then y is in F, and x,y in F with $d(x,y) < \delta_2$ implies that $d(\pi^1(x), \pi^1(y)) < \delta_o$.

Also there exists $\delta_3 > 0$ such that $d(x,y) < \delta_3$ implies that $d(\phi_t x, \phi_t y) < \min\{\delta_1, \delta_2\}$, for all t, $|t| \leq 1$.

If x and y are in X, then x and y have a unique representation as $x = \phi_{s_o} x_o$, $y = \phi_{t_o} y_o$ where x_o and y_o are in K, and $0 \leq s_o, t_o < 1$.

Moreover if $\varepsilon > 0$ is given, there exists $\delta_4 > 0$ such that $d(x,y) < \delta_4$ implies that $|s-t| < \varepsilon$.

Choose $\delta = \min\{\delta_3, \delta_4\}$, and suppose $d(\phi_t x, \phi_{\alpha(t)} y) < \delta$ for all t in $\underline{R}$.

Then $d(\phi_t(\phi_{-s_o}x), \phi_{\alpha(t)}(\phi_{-s_o}y)) < \min\{\delta_1, \delta_2\}$.

$\therefore$ $d(\phi_n x_o, \phi_{\alpha(n)}(\phi_{-s_o}y)) < \delta_2$ for all integers n

$\therefore$ $d(h^n x_o, h^n y_o) < \delta_o$ for all integers n .

$\therefore$ $y_o = x_o$,

and therefore $y = \phi_p x$ where $|p| < \varepsilon$.

Hence (X,ϕ) is expansive.

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§3.3 Factors of Expansive Flows.

Let (X,ϕ) and (Y,ψ) be continuous flows and suppose we have a continuous surjection $p : X \to Y$ mapping ϕ-orbits onto ψ-orbits. Then we shall say (Y,ψ) is a factor (via p) of (X,ϕ). It is not true in general that a factor of an expansive flow is expansive, since the corresponding result is false for expansive homeomorphisms. We simply take a discrete counterexample (See e.g. [Walters 1]) and suspend it to get a counterexample for the continuous case.

However we do have the following result.

Theorem 3.31. Let (X,ϕ) and (Y,ψ) be compact continuous flows and suppose (Y,ψ) is a factor (via p) of (X,ϕ). If p is a local homeomorphism, then (X,ϕ) is expansive if and only if (Y,ψ) is expansive.

Proof. We note that, if the fixed point set of (X,ϕ), F_ϕ say, is discrete, then so is F_ψ, the fixed point set of (Y,ψ), and vice versa. We therefore assume that both F_ϕ and F_ψ are discrete.

Now since p is a local homeomorphism, by compactness there exist δ_X and $\delta_Y > 0$ such that for all x in X, p maps $B(x,\delta_X)$ homeomorphically onto $B(px, \delta_{px})$ where $\delta_{px} \geq \delta_Y$.

Also there exist μ, $\mu' > 0$ such that each point y in $Y \setminus F_\psi$ has a μ'-flow box neighbourhood $F'_y \subset B(y, \delta_Y)$, each point x in $X \setminus F_\phi$ has a μ-flow box neighbourhood $F_x \subset B(x,\delta_x)$, and $pF_x \subset F'_{px}$.

Now the proof of Lemma 3.11 is essentially a local proof; the map α was constructed firstly with domain $[-\mu,\mu] \times X$, and then the domain of α extended to $\underline{R} \times X$ using the flow properties of ϕ and ψ.

Hence the same argument will apply here, since $p|F_x$ is a homeomorphism, mapping ϕ-orbits to ψ-orbits.

Therefore, as in Lemma 2.11, we have a continuous map $\alpha : \underline{R} \times X \to \underline{R}$ with α_x in hom$(\underline{R})$ for each x in X and $p\, \phi_t x = \phi_{\alpha(t,x)}\, px$, for all t in $\underline{R}$, for all x in X.

If $\varepsilon > 0$ is given, there exists $\delta > 0$ such that if x and y are in X and $y = \phi_t(x)$, for some t with $|t| < \delta$, then

$py = \psi_{t'}(px)$, where $|t'| < \varepsilon$.

Now suppose (X,ϕ) is expansive. Let $\varepsilon > 0$ be given, and choose $\varepsilon' = \delta$ satisfying the statement above. Correspondingly, there exists $\delta' > 0$ such that if x and y are in X, and β, γ are in hom($\underline{R}$) with $d(\phi_{\beta(t)}(x), \phi_{\gamma(t)}(y)) < \delta'$ for all t in R, then $y = \phi_p x$, $|p| < \varepsilon'$.

Also p is a local homeomorphism, hence there exists $\delta_1 > 0$ such that if y_1 and y_2 are in Y, and $d(y_1, y_2) < \delta_1$, and if we choose x_1 in X with $px_1 = y_1$, and consider the unique x_2 in $B(x, \delta_X)$ with $px_2 = y_2$, then $d(x_1, x_2) < \delta'$.

Let y_1 and y_2 be in Y, and ξ, ζ be in hom($\underline{R}$) with $d(\psi_{\xi(t)}(y_1), \psi_{\zeta(t)}(y_2)) < \delta_1$, for all t in R.

Choose x_1 in $p^{-1}\{y_0\}$, and define x_2 as the unique point in $p^{-1}\{y_0\}$ with $d(x_1, x_2) < \delta'$

Then ξ and ζ induce β and γ in hom($\underline{R}$) with $d(\phi_{\beta(t)}(x_1), \phi_{\gamma(t)}(x_2)) < \delta'$ for all t in $\underline{R}$,

$\therefore$ $x_2 = \phi_r x_1$ where $|r| < \varepsilon'$.

$\therefore$ $px_2 = \psi_s px_1$, where $|s| < \varepsilon$.

Thus (X,ϕ) expansive implies that (Y,ψ) is expansive.

Conversely, if (X,ϕ) is not expansive, then neither is (Y,ψ). For (since p is a local homeomorphism), there exists $\varepsilon_x > 0$, such that for all ε with $0 < \varepsilon \leq \varepsilon_x$, there exists $\varepsilon' > 0$ such that if x and y are in X, and $d(x,y) < \delta_X$, then $[y \neq \phi_\tau x, |\tau| < \varepsilon]$ implies that $py \neq \psi_t\, px$, for any t with $|t| < \varepsilon'$.

Since (X,ϕ) is not expansive, there exists $\varepsilon_1 > 0$ such that for all $\delta > 0$, there are x and y in X, α and β in hom $(\underline{R})$, with $y \neq \phi_\tau x$, $|\tau| < \varepsilon_1$, and $d(\phi_{\alpha(t)}\, x, \phi_{\beta(t)} y) < \delta$ for all t in $\underline{R}$.

Choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_x\}$ and let ε' be the corresponding ε' above.

If $\delta > 0$ is given, there exists $\delta_1 > 0$ with $d(x,y) < \delta_1$ implying that $d(px,py) < \delta$.

Let $\delta_2 = \min\{\delta_1, \delta_X\}$.

Take x and y in X and α, β in hom$(\underline{R})$ with $y \neq \phi_\tau x$ for any τ with $|\tau| < \varepsilon$, and $d(\phi_{\alpha(t)}(x), \phi_{\beta(t)}(y)) < \delta_2$ for all t in $\underline{R}$.

Then

$$d(p\phi_{\alpha(t)}x, \ p\phi_{\beta(t)}y) < \delta \text{ for all } t \text{ in } \underline{R}$$

i.e. $d(\psi_{\xi(t)}px, \psi_{\zeta(t)}py) < \delta$ for all t in $\underline{R}$ and some ξ and ζ in hom$(\underline{R})$.

But $py \neq \psi_t px$ for any t with $|t| < \varepsilon'$.

Thus (Y,ψ) is not expansive.

<u>Corollary 3.32</u>. It is known (see e.g. [Lima 1]) that if Y is a covering space of X, a continuous flow ϕ on X lifts to a continuous flow ψ on Y. Thus if X and Y are compact, then (X,ϕ) is expansive if and only if the lifted flow (Y,ψ) is expansive.

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§3.4 Commuting Maps and Flows

If h is an expansive homeomorphism on a compact space X, then it is known that there are only a countable number of distinct continuous maps commuting with h, i.e. maps $\psi : X \to X$ such that $h\psi = \psi h$. (See e.g. [Ping-Fun Lam 1]). A corollary of this is that an expansive homeomorphism on a compact space cannot be embedded in a continuous flow. We consider now some analogous results for continuous flows.

Let ϕ be a continuous flow on a compact space X and let $H(X)$ denote the (separable) metric space of homeomorphisms of X, with metric given by $d(h,g) = \sup\limits_{x\in X} d(fx, gx)$.

Consider the subspace $H_o(X) \subset H(X)$ consisting of those homeomorphisms h which map ϕ-orbits to ϕ-orbits. Then, for h in $H_o(X)$, there exists a time change flow ψ of ϕ such that, for all x in X, $h\, \phi_t(x) = \psi_t(hx)$.

We shall say f and g in $H_o(X)$ are equivalent, if, for all x in X, $O_\phi(fx) = O_\phi(gx)$.

<u>Proposition 3.41</u>. If (X,ϕ) is expansive, there exists $\delta > 0$, such that, for all f and g in $H_o(X)$, $d(f,g) < \delta$ implies that f and g are equivalent.

<u>Proof</u>. There exists $\delta > 0$ such that for x,y in X, α,β in hom($\underline{R}$), $d(\phi_{\alpha(t)}(x), \phi_{\beta(t)}(y)) < \delta$ for all t in $\underline{R}$ implies that $y = \phi_p x$, $|p| \leq 1$.

Suppose f,g are in $H_o(X)$ and $d(f,g) < \delta$. Thus for all x in X, and for all t in $\underline{R}$,

$$d(f\,\phi_t(x), \; g\phi_t(x)) < \delta .$$

i.e. there are time changes ψ and $\tilde{\psi}$ of ϕ with $d(\psi_t(fx), \tilde{\psi}_t(gx)) < \delta$ for all t in $\underline{R}$ i.e. $d(\phi_{\alpha(t)}(fx), \phi_{\beta(t)}(gx)) < \delta$ for all t in $\underline{R}$ where α and β are in hom($\underline{R}$).

Thus, for all x in X, $gx = \phi_p(fx)$ for some p with $|p| \leq 1$.

<u>Commuting Flows</u>.

Recall that if ϕ and ψ are two flows on X, ϕ and ψ are said to <u>commute</u> if for all x in X, and for all s and t in $\underline{R}$,

$$\phi_s \, \psi_t(x) = \psi_t \, \phi_s(x)$$

<u>Definition 3.42</u>. We shall say ϕ and ψ <u>commute weakly</u> if there exist time change flows $\tilde{\phi}$ of ϕ and $\tilde{\psi}$ of ψ such that $\tilde{\phi}$ and $\tilde{\psi}$ commute.

Clearly, if ϕ and ψ commute, they commute weakly.

<u>Theorem 3.43</u>. Let ϕ be an expansive flow on X (compact). Then a flow ψ on X commutes weakly with ϕ if and only if ψ is a time change flow of ϕ (up to orientation) and therefore ψ commutes with ϕ if and only if ψ commutes weakly with ϕ.

<u>Proof</u>. As above, there exists $\delta > 0$ such that if x and y are in X, α and β are in hom($\underline{R}$), and $d(\phi_{\alpha(t)}(x), \phi_{\beta(t)}(y)) < \delta$ for all t in $\underline{R}$, then $y = \phi_p x$, where $|p| \leq 1$.

Now suppose ψ commutes weakly with ϕ. There exists $\mu > 0$ such that $|t| \leq \mu$ implies that

$d(\tilde{\psi}_t, id) < \delta$ where $\tilde{\psi}$ is as in Definition 3.42.

Therefore, for all x in X, and for all s in $\underline{R}$,

$d(\tilde{\psi}_t \tilde{\phi}_s(x), \tilde{\phi}_s(x)) < \delta$ for t with $|t| \leq \mu$

i.e. $d(\tilde{\phi}_s \tilde{\psi}_t(x), \tilde{\phi}_s(x)) < \delta$.

Therefore by expansiveness of ϕ, for all x in X and for t with $|t| \leq \mu$, $\tilde{\psi}_t(x) = \phi_{p(t)}(x)$ where $|p(t)| \leq 1$.

Therefore the orbits of ψ are the same as the orbits of ϕ, and hence, up to orientation of ψ, ψ is a time change flow of ϕ.

Thus if ψ commutes weakly with ϕ, then ψ is a time change flow of ϕ(up to orientation), and therefore ψ commutes with ϕ.

Conversely a time change flow ψ of ϕ commutes with ϕ.

§3.5 Periodic Orbits.

We recall, that for an expansive homeomorphism h on a compact space X, the number of periodic points of h of period p is less than or equal to n^p, where n is the number of elements in a generator for (X,h). We now prove the analogue of this result for expansive flows.

Theorem 3.51. Let (X,ϕ) be an expansive continuous compact flow. Let $\mathcal{F}$ be a μ-generator for $X \setminus F_\phi$, (where F_ϕ denotes the fixed point set of ϕ), consisting of p elements. Then, for fixed $T > 0$, the number of periodic orbits with period $\leq T$ is bounded above by $p^{[T/\mu]}$

where [ ] means "integer part of".

Proof. Let x be in X. Then x admits a sequence (u_i)

with $\phi_{t_o} x$ in S_{u_o}, $0 \leq |t_o| < \mu$

and $\phi_{t_i} x$ in S_{u_i}, $\mu < t_i - t_{i-1} < 3\mu$

If x is periodic, with period $\leq T$, then x admits such a sequence (u_i) which is periodic with period $\leq [T/\mu] - 1$.

Moreover, if x and y in X are on different periodic orbits, then x and y cannot admit the same sequences since $\mathcal{F}$ is a generator. The maximum possible number of distinct sequences of period m is p^m. Hence the maximum possible number of distinct periodic orbits of period $\leq T$ is given by

$$\sum_{m=1}^{m=[T/\mu]-1} \frac{p^m}{m}$$, and this number is less than $p^{[T/\mu]}$.

<u>Periodic Orbits for Weakly Expansive Flows.</u>

We show that there is a slightly weaker version of this result for weakly expansive flows.

<u>Theorem 3.52.</u> Let ϕ be a weakly expansive flow on a compact space X. Then, for fixed $T > 0$, the number of periodic orbits of ϕ of period $\leq T$, is finite.

<u>Proof.</u> Suppose there were an infinite number of orbits of period $\leq T$. Then, by compactness of $[0,T]$, there exists T_0 in $[0,T]$ such that, for all $\delta_1 > 0$, there are an infinite number of periodic orbits with period in the range $(T_0 - \delta_1, T_0 + \delta_1)$

Let $\delta > 0$ be given. There exist δ_2 and $\delta_3 > 0$, such that $d(x,y) < \delta_2$ and $|s - t| < \delta_3$ implies that

$d(\phi_t(x), \phi_s(y)) < \delta$ for t with $0 \leq t \leq T_0$.

By compactness of X, we can select x and y in X with $d(x,y) < \delta_2$, x and y on different orbits, and such that x and y are periodic with periods in the range $(T_0 - {}^{\delta_3}/_2 , T_0 + {}^{\delta_3}/_2)$.

We construct a time change flow ψ of ϕ under which x and y are both periodic of period T_0, and such that, for all t in $\underline{R}$, $d(\psi_t x, \psi_t y) < \delta$, which contradicts the hypothesis of weak expansiveness of ϕ.

Let x be periodic of period $T_o - \Delta$

and y be periodic of period $T_o + \Theta$.

Define $\lambda : X \to \underline{R}$ as follows:

For $z \in O(x)$, $\lambda(z) = 1 - \frac{\Delta}{T_o}$.

For $z \in O(y)$, $\lambda(z) = 1 + \frac{\Theta}{T_o}$, and

then extend the domain of definition of λ to all of X using the Tietze extension theorem, ensuring that λ remains strictly positive.

Define ψ_t by $\psi_t(x) = \phi_{\alpha(t,x)}(x)$, where $\alpha(t,x)$ is the unique solution of

$$t = \int_0^{\alpha(t,x)} \frac{ds}{\lambda . \phi_s(x)}$$

It is easy to check that x and y are periodic under ψ of period T_o, and that $d(\psi_t x, \psi_t y) < \delta$ for all t in $\underline{R}$.

§4. Asymptotic Properties of Expansive Flows

In this section we show that expansive flows exhibit many of the asymptotic properties of expansive homeomorphisms:- in particular we show that non-isolated closed orbits of an expansive flow are topologically sources, saddles or sinks. The development of this section parallels that in Bryant and Walters' paper in the discrete case. (See [Bryant and Walters 1]).

§4.1. Asymptotic Orbits.

Definition 4.11. Let $\mathcal{F} = \{\overset{\circ}{F}_i\ (S_i,\mu)\}$ be a μ flow box cover for a continuous flow ϕ on X. We say x in X admits a block of length 2N + 1 if there exists an N-truncated sequence $u_{-N}, \ldots u_o, \ldots u_N$ with $\phi_{t_i} x$ in S_{u_i}, $-N \leq i \leq N$, where $|t_o| \leq \mu$, $\mu/_4 \leq t_i - t_{i-1} \leq 3\mu$.

Theorem 4.12. Let (X,ϕ) be expansive with μ-generator $\mathcal{F}$. For each non-negative integer N, there exists $\varepsilon > 0$ such that, for x,y in X, $d(x,y) < \varepsilon$ implies that x and y admit a common block of length 2N + 1.

Conversely, for all $\varepsilon > 0$, there exists a positive integer N such that if x and y in X admit a common block of length 2N + 1, then $d(\pi_o^{-1}(x), \pi_o^{-1}(y)) < \varepsilon$.

Proof. Let N be a fixed non-negative integer. For all $\zeta > 0$, there exists $\delta > 0$ such that for x,y in X with $d(x,y) < \delta$, $d(\phi_t x, \phi_t y) < \zeta$ for all t with $|t| \leq N.3\mu$.

As in the proof of Theorem 2.23, we choose ζ sufficiently small so that

(1) x and y admit a common N-truncated sequence of flow boxes (using the Lebesgue no of the cover $\mathcal{F}$),

and (2) The times of successive intersections of $O(x)$ and $O(y)$ with the local sections satisfy the required condition.

Then we take the corresponding $\delta > 0$ for our ε in the first part of the theorem.

Conversely, let $\varepsilon > 0$ be given. If the conclusion did not hold, we would have the following situation.

For each $j > 0$, there exist x^j and y^j in X and sections $S_{j,i}$, $-j \leq i \leq j$, with

$$\left.\begin{matrix} \phi_{t_{j,i}} x^j \\ \\ \phi_{s_{j,i}} y^j \end{matrix}\right\} \text{ in } S_{j,i} \left(\begin{matrix} |t_{j,o}|, \quad |s_{j,o}| \leq \mu \\ \\ \mu/_4 \leq t_{j,i+1} - t_{j,i} \leq 3\mu \\ \\ \mu/_4 \leq s_{j,i+1} - s_{j,i} \leq 3\mu \end{matrix} \right)$$

and $d(\pi^{1}_{j,o}(x^j), \ \pi^{1}_{j,o}(y^j)) \geq \varepsilon \quad$ —(1).

Firstly we may suppose $x^j \to x$, $y^j \to y$, with x and y in X. We shall show that x and y admit a common sequence of sections, thus contradicting the fact that $\mathcal{F}$ is a generator.

Now an infinite number of the $S_{j,o}$ are the same, say equal to S_o, (since $\mathcal{F}$ has only a finite number of elements).

$$\therefore \left.\begin{array}{l} \phi_{t_{j,o}} x^j \\ \\ \phi_{s_{j,o}} y^j \end{array}\right\} \text{are in } S_o \text{ for infinitely many } j.$$

By passing to a subsequence if necessary, we may suppose that $\phi_{t_{j,o}} x^j \to \phi_{t_o} x = x_o$ say,

$$\phi_{s_{j,o}} y^j \to \phi_{s_o} y = y_o \text{ say},$$

$$\text{where } |s_o|, \ |t_o| \leq \mu ,$$

x_o and y_o are in S_o, and $x_o \neq y_o$, by (1).

(The sequence of points $\phi_{t_{j,o}} x^j$ converges to a point on the orbit of x, since the flow box structure of $F_o = S_o \times [-\mu.\mu]$ ensures that $\{t_{j,o}\}$ approaches a unique limit to as $j \to \infty$).

Now consider a fixed $i > 0$, and suppose that S_i, $x_i = \phi_{t_i} x$, $y_i = \phi_{s_i} y$ have been defined with

$$\phi_{t_{j,i}} x^j \to x_i$$

$$\phi_{s_{j,i}} y^j \to y_i$$

As before, an infinite number of the $S_{j,\ i+1}$ are the same, say equal to S_{i+1}, and again we can show that

$$\phi_{t_{j,i+1}}(x^j) \to \phi_{t_{i+1}}(x) \qquad \text{with} \quad {}^{\mu}/_{4} \le t_{i+1} - t_i \le 3\mu$$

$$\phi_{s_{j,i+1}}(y^j) \to \phi_{s_{i+1}}(y) \qquad {}^{\mu}/_{4} \le s_{i+1} - s_i \le 3\mu$$

In a similar manner, if, for a fixed $i \le 0$, S_i, $x_i = \phi_{t_i} x$, $y_i = \phi_{s_i} y$, have been defined, we can inductively define S_{i-1}, x_{i-1} and y_{i-1}.

Thus we have shown that x and y admit a common sequence of sections.

Definition 4.13. Let (X,ϕ) be a continuous flow. If x and y are in X, we say that x and y (or more precisely $O(x)$ and $O(y)$) are weakly positively asymptotic if there exists α in hom($\underline{R}$) such that, for all $\varepsilon > 0$, there exists T in $\underline{R}$ such that, for all $t > T$, $d(\phi_t x, \phi_{\alpha(t)} y) < \varepsilon$.

We define weak negative asymptoticity in a similar way, replacing $t > T$ by $t < T$ in the statement above.

Definition 4.14. We say that x and y in X are positively [negatively] asymptotic if there exists a time change flow ψ of ϕ such that, for all $\varepsilon > 0$, there exists T in $\underline{R}$ with $d(\psi_t x, \psi_t y) < \varepsilon$, for all $t > T$ [$t < T$]. (We say that $O(x)$ and $O(y)$ are positively [negatively] asymptotic if there exist P,Q in $\underline{R}$ with $\psi_P(x)$, $\psi_Q(y)$ positively [negatively] asymptotic.

Remarks. Clearly if x and y are asymptotic, they are weakly asymptotic. The two concepts of asymptoticity are related to expansiveness and weak expansiveness. If we could show that a weakly expansive flow was expansive, presumably we could show that two weakly asymptotic orbits were in fact asymptotic. It follows from a similar proof to that of Theorem 3.28, that if (X,ϕ) admits a global section, then the two concepts are equivalent i.e. two orbits of (X,ϕ) are asymptotic if and only if they are weakly asymptotic.

Definition 4.15. Let (X,ϕ) be a continuous flow with μ flow box cover $\mathcal{F}$. If x is in X, we say x admits a positive sequence (u_i) for each non negative integer i, $\phi_{t_i}x$ is in S_{u_i} where $0 \leq |t_o| \leq \mu$ and $\mu/4 \leq t_i - t_{i-1} \leq 3\mu$.

We have a similar definition for "x admitting a negative sequence".

Theorem 4.16. Let (X,ϕ) be an expansive flow with μ-generator $\mathcal{F}$. If x and y are in X, $O(x)$ and $O(y)$ are weakly positively asymptotic if and only if there exist S and T in $\underline{R}$ such that $\phi_T(x)$, $\phi_S(y)$ admit a common positive sequence of local sections.

Proof. Suppose $O(x)$ and $O(y)$ are positively asymptotic. As in the proof of Theorem 2.23, we can choose $\delta > 0$ sufficiently small so that if z,w are in X and there exists β in hom($\underline{R}$) with

$d(\phi_t(z), \phi_{\beta(t)}(w)) < \delta$ for all t in $\underline{R}^+$, then z,w admit a common positive sequence of local sections. Since O(x) and O(y) are weakly positively asymptotic, there exists α in hom($\underline{R}$) such that for all t > some T in $\underline{R}$, $d(\phi_t(x), \phi_{\alpha(t)}(y)) < \delta$. Hence, taking $z = \phi_T(x)$, $w = \phi_{\alpha(T)}(y)$, and β in hom($\underline{R}$) defined by $\beta(t) = \alpha(T + t) - \alpha(T)$ gives the required result.

Conversely suppose there exist T,S in $\underline{R}$ with $\phi_T(x)$, $\phi_S(y)$ admitting a common positive sequence of local sections (u_i).

i.e. We have (positive) sequences of times (t_i) and (s_i)

with $\begin{matrix}\phi_{t_i+T}(x)\\ \phi_{s_i+S}(y)\end{matrix}$ in S_{u_i} , $\begin{aligned}&t_o = s_o = 0.\\ &\mu/_4 \leq t_{i+1} - t_i \leq 3\mu .\\ &\mu/_4 \leq s_{i+1} - s_i \leq 3\mu .\end{aligned}$

Let $\varepsilon > 0$ be given .

There exist δ_1, $\zeta > 0$ such that, for z,w in X, $d(z,w) < \delta_1$ and $|s - t| < \zeta$ imply that

$d(\phi_t(z), \phi_s(w)) < \varepsilon$, for $0 \leq s,t, \leq 3\mu$.

There exists $\delta_2 > 0$, such that for all flow boxes F with $\overset{\circ}{F}$ in $\mathcal{F}$, z,w in F and $d(z,w) < \delta_2$ implies that

$$|\pi^2_F(z) - \pi^2_F(w)| < \zeta.$$

Take $\delta = \min\{\delta_1, \delta_2\}$.

By the previous theorem, there exists a positive integer N such that if z,w admit a common block of length 2N + 1 then

$d(\pi_o^1(z), \pi_o^1(w)) < \delta$. Thus for $i \geq N$, $d(\phi_{t_i+T}(x), \phi_{s_i+S}(y)) < \delta$.

Define α in hom($\underline{R}$) as follows:

$\alpha(t) = t$ for $t \leq 0$.

For t in $(0, t_N + T)$, $\alpha(t) = (\frac{t}{t_N + T})(S_N + S)$.

For $t \geq t_N + T$, define α piece-wise as follows:

For each $i \geq N$, map $[t_i + T, \ t_{i+1} + T]$ linearly

onto $[s_i + S, \ s_{i+1} + S]$.

Then by the inequalities above, for $t \geq t_N + T$, $d(\phi_t x, \phi_{\alpha(t)} y) < \varepsilon$.

Thus $O(x)$ and $O(y)$ are weakly positively asymptotic.

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§4.2 Existence of Asymptotic Orbits.

It is known, (see e.g. [Bryant and Walters 1]) that if h is an expansive homeomorphism on a compact infinite space X, then (X,h) has a pair of positively asymptotic points, and a pair of negatively asymptotic points. We have only been able to prove a partial analogue for continuous expansive flows.

Theorem 4.21. Let (X,ϕ) be a compact expansive flow with an infinite number of distinct orbits. Then there exists a pair of points in X which are weakly asymptotic in one sense.

Proof. Let $\mathcal{F} = \{\overset{\circ}{F_i}(S_i,\mu)\}$ be a μ-generator for (X,ϕ). There exists a local section, S_o say, containing points from an infinite number of distinct orbits. Select x_o and y_o in S_o on different orbits. Either x_o and y_o admit a common positive sequence of local sections, in which case x_o and y_o are weakly positively asymptotic, or they do not. If the second possibility always occurs, we can select sequences of points (x_i) and (y_i) in S_o with $d(x_i, y_i) \to 0$ and x_i, y_i admitting a common positive block of local sections of length N_i, where $N_i \to \infty$ as $i \to \infty$. Without going through horrific mechanics similar to those in the proof of Theorem 4.12, we assert that the statement above implies the existence of two distinct points x_o and y_o in X, which admit a negative sequence of local sections. Thus x_o and y_o are weakly negatively asymptotic.

Theorem 4.22. If (X,ϕ) is an expansive continuous compact flow admitting a global section, then (X,ϕ) admits a pair of positively asymptotic orbits, and a pair of negatively asymptotic orbits.

Proof. By the remark at the beginning of this section, Lemma 3.27, Theorem 3.28 and the following

Note. Let h be an expansive homeomorphism on a compact space X, and suppose x, $h^p x$ in X are positively asymptotic with respect to h.

i.e. $d(h^i x, h^{i+p} x) \to 0$ as $i \to \infty$. Consider z in $\omega(x)$, and let (m_i) be an increasing sequence of integers with $h^{m_i} x \to z$. Then $h^{m_i+p}(x) \to z$ and hence $h^{-p}(z) = z$. Thus $O(z)$ is a non-isolated periodic orbit which is a saddle or sink in the terminology of [Bryant and Walters], and therefore there exist points in X on different h orbits which are positively asymptotic.

---

§4.3 Asymptotic Properties Near Periodic Orbits.

Let (X,ϕ) be a continuous flow and let x_o be in X with $O(x_o)$ a non-isolated periodic orbit. Let $S(x_o)$ be a local section of the flow containing x_o.

If $\overline{B(x_o,\delta)}$ is the closed δ-ball with centre x_o and radius δ, there exists δ sufficiently small so that for x in $\bar{U} = B(x_o,\delta) \cap S(x_o)$, there is a well defined map $h: \bar{U} \to S(x_o)$ which sends u in $\bar{U}$ onto $\phi_{t_u}(u)$ where t_u is the time of first return of u to $S(x_o)$. h is a homeomorphism onto its image and $h(x_o) = x_o$.

Lemma 4.31. Let (X,ϕ) be a continuous compact flow, with x_o a non-isolated periodic point of ϕ of period T_o, and let $S(x_o)$, $\bar{U}$, h be constructed as above. Then there is a time change flow ψ of ϕ such that, for all u in $\bar{U}$, $\psi_{T_o}(u) = h(u)$.

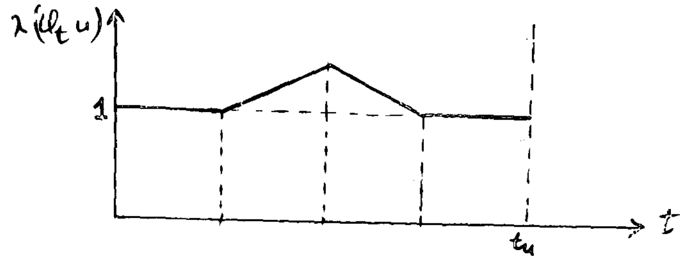
<u>Proof</u>. Consider u in $\bar{U}$. Suppose $h(u) = \phi_{t_u}(u)$ with $t_u = T_o + \Delta_u$ say. We define λ from $\{\phi_t u : 0 \leq t \leq t_u\}$ to $\underline{R}^+$ as follows:

For $0 \leq t \leq \frac{t_u}{4}$, $\lambda(\phi_t u) = 1$.

For $\frac{t_u}{4} \leq t \leq \frac{t_u}{2}$, $\lambda(\phi_t u) = \frac{16\,\Delta_u}{t_u^{\,2}} \cdot t - \frac{4\Delta_u}{t_u} + 1$.

For $\frac{t_u}{2} \leq t \leq 3\frac{t_u}{4}$, $\lambda(\phi_t u) = \frac{-16\Delta_u}{t_u^{\,2}} \cdot t + \frac{12\Delta_u}{t_u} + 1$.

For $3\frac{t_u}{4} \leq t \leq 1$, $\lambda(\phi_t u) = 1$.



The manner of construction of λ ensures that λ is continuous on the closed subspace $\bigcup_{u \text{ in } \bar{U}} \{\phi_t u : 0 \leq t \leq t_u\}$, and we extend the domain of definition of λ to X.

Then define ψ_t by

$\psi_t(x) = \phi_{h(t,x)}(x)$, where $h(t,x)$ is the unique solution to the equation $t = \int_o^{h(t,x)} \frac{ds}{\lambda\phi_s(x)}$

In particular we see that, for u in $\bar{U}$, $h(T_o, u) = T_o + \Delta_u$, and hence $\psi_{T_o}(u) = h(u)$.

Corollary 4.32. If (X,ϕ) is a continuous compact flow and $O(x_o)$, $O(y_o)$ are weakly positively [negatively] asymptotic with $O(x_o)$ periodic, then $O(x_o)$, $O(y_o)$ are positively [negatively] asymptotic.

Lemma 4.33. Let (X,ϕ) be an expansive continuous compact flow, and let $O(x_o)$ be a non-isolated periodic orbit. If $S(x_o)$, $\bar{U}$, and h are as described above, then the map h is a local expansive homeomorphism; i.e. there is a relatively open neighbourhood $V \subset \bar{U}$ of x_o, such that for $x \neq x_o$ in V, there exists an integer n such that $h^n x$ is not in V (or is not defined).

Proof. If the assertion were false, then for all $\delta > 0$, we could find $x \neq x_o$ in V, with $d(\psi_t x, \psi_t x_o) < \delta$ for all t in $\underline{R}$, where ψ is the time change flow of ϕ constructed in Lemma 4.31, thus contradicting expansiveness of ϕ.

Corollary 4.34. The analysis for expansive homeomorphisms of asymptotic properties near periodic orbits carries over completely to the continuous case to give us

Theorem 4.35. Let (X,ϕ) be an expansive continuous compact flow. Each non-isolated periodic orbit, $O(x_o)$ say, is one of the following mutually exclusive types.

<u>Type 1</u> (Sinks). There exists an open neighbourhood U of $O(x_o)$ such that $\phi_t U \subset U$ for all t in $\underline{R}^+$, and $\bigcap_{t\geq o} \phi_t U = O(x_o)$.

<u>Type 2</u> (Sources). There exists an open neighbourhood U of $O(x_o)$ such that $\phi_t U \subset U$ for all t in $\underline{R}^-$ and $\bigcap_{t\leq o} \phi_t U = O(x_o)$.

<u>Type 3</u> (Saddles). There exist p and q in X with O(p) positively asymptotic to $O(x_o)$, and

O(q) negatively asymptotic to $O(x_o)$.

<u>Corollary</u>. If (X,ϕ) is a compact expansive flow which has a non-isolated periodic orbit, then there is a time change flow ψ of ϕ which is <u>non-distal</u>. We also have a direct analogue of Theorem 10 in [Bryant and Walters 1], viz

<u>Theorem 4.36</u>. Let (X,ϕ) be an expansive compact flow. If (X,ϕ) has a dense positive semi-orbit then all of the non-isolated periodic orbits are of Type 3.

<u>Proof</u>. Suppose $O(x_o)$ is a periodic orbit of Type 1. Let $O^+(y_o)$ be the dense positive semi-orbit. There is an open neighbourhood U of $O(x_o)$ such that $\phi_t U \subset U$ for all $t \geq 0$ and $\bigcap_o^\infty \phi_t U = O(x_o)$.

Let V be an open neighbourhood of $O(x_o)$ with $\bar{V} \subset U$. Since $O(y_o)$ is positively asymptotic to $O(x_o)$, there exists $T > 0$, such that for $t \geq 0$, $\phi_t\, y_o$ is in the complement of $\bar{V}$ only for $0 \leq t < T$. In the period $[0,T)$, $0^+(y_o)$ has only a finite number of points of intersection with $\bigcup_i S_i \cap \bar{V}^c$, where the S_i are local sections of a generator for ϕ.

However, for some $s > 0$, y_o is not in $\phi_s U$, i.e. $\phi_{-s}(y_o)$ is not in U.

Therefore, for all $t \geq 0$, $\phi_{-t-s}(y_o)$ is not in $\phi_{-t} U \supset U$.

Thus the whole negative semi-orbit of $\phi_{-s}(y_o)$ lies outside $\bar{V}$ and therefore, since $0(y_o)$ is not periodic, has an infinite number of points of intersection with $\bigcup_i S_i \cap \bar{V}^c$, thus contradicting the density of $0^+(y_o)$.

Similarly $0(x_o)$ cannot be of Type 2.

<u>Lemma 4.37</u>. Let (X,ϕ) be a compact continuous flow. If the non-wandering set $\Omega(\phi)$ of (X,ϕ) consists of a finite set of fixed points, together with a finite set of periodic orbits, then every orbit of (X,ϕ) is positively (and negatively) asymptotic to either a fixed point or a periodic orbit.

<u>Proof</u>. Suppose $\Omega(\phi) = E_1 \cup E_2 \cup \ldots \cup E_p$ where each E_i is either a fixed point or a periodic orbit. Surround each E_i with an open neighbourhood V_i such that, for $i \neq j$, $V_i \cap V_j = \emptyset$.

Suppose $O(x)$ is not positively asymptotic to any of the E_j. Then since $O(x)$ is connected, there exist times t_j with $t_j \to \infty$ and $\phi_{t_j} x$ not in $\bigcup_i V_i$, which gives us a contradiction.

Corollary 4.38. Let (X,ϕ) be a compact expansive flow with an uncountable number of distinct orbits. Then $\Omega(\phi)$ consists of more than a finite set of fixed points and periodic orbits. (See Lemma 6.2 for a detailed proof).

§5. A Lifting Theorem

Definition 5.1. The sequence space on k symbols is defined to be the space $\sum^k$ of all bisequences $\underline{u} = (u_i)$ where each entry u_i of $\underline{u}$ is from the set $Z_k = \{0,1 \ldots k-1\}$. $\sum^k$ is a compact metric space, a metric being given by $d(\underline{u},\underline{v}) = \sum_i \frac{1}{2^{|i|}} d(u_i,v_i)$

The shift map σ on $\sum^k$ is defined by $\sigma((u_i)) = (u_{i-1})$.

If h is an expansive homeomorphism on a compact space X, then it is known, (see e.g. [Keynes and Robertson 1]), that (X,h) is a factor of $(\sum_o{}^k, \sigma)$ where $\sum_o{}^k$ is a closed shift invariant subspace of $\sum^k$, for some $k > 0$.

We now prove the analogous result for flows.

Let (X,ϕ) be a continuous compact fixed point free flow which is expansive with μ-generator $\mathcal{F} = \{\overset{o}{F}_1(S_1,\mu),\ldots \overset{o}{F}_p(S_p,\mu)\}$.

Recall that x in X admits a sequence (u_i) if there exists a sequence of times t_i with $\mu/4 \leq t_i - t_{i-1} \leq 3\mu$, and $|t_o| \leq \mu$, such that, for each integer i, $\phi_{t_i} x$ is in S_{u_i}.

Define $\sum_o{}^p$ to be the subset of $\sum^p$ consisting of sequences (u_i) such that there exists x in X which admits (u_i).

Clearly $\sum_o{}^p$ is a shift invariant subspace of $\sum^p$.

Lemma 5.2. $\sum_o{}^p$ is a closed subspace of $\sum^p$.

Proof. We have to show that if $\underline{u}_j \to \underline{u}$ and each $\underline{u}_j$ is in $\sum_o^p$, then so is $\underline{u}$.

If $\underline{u}_j \to \underline{u}$, then for all integers i, there exists j_i such that for $j \geq j_i$, $u_{i,j} = u_i$ where $u_{i,j} = i^{th}$ place in the sequence $\underline{u}_j$,

$u_i = i^{th}$ place in the sequence $\underline{u}$.

For each j, $\underline{u}_j$ is in $\sum_o^p$,

i.e. for each positive integer j, there exist points x_j in X and an i-sequence of times $t_{i,j}$ with $|t_{o,j}| \leq \mu$,

$\mu/4 \leq t_{i,j} - t_{i-1,j} \leq 3\mu$, and $\phi_{t_{i,j}} x_j = (x_j)_i$ say in $S_{u_{i,j}}$.

Consider a fixed i. For $j \geq j_i$, $(x_j)_i$ is in S_{u_i}.

S_{u_i} is closed. $\therefore$ by taking a convergent subsequence if necessary, we may assume that $(x_j)_i$ converges to x_i, say, in S_{u_i}.

Similarly, for $j \geq j_{i+1}$,

$$(x_j)_{i+1} \to x_{i+1} \text{ in } S_{u_{i+1}} .$$

But $(x_j)_{i+1} = \phi_{p_{i,j}} (x_j)_i$ where $\mu/4 \leq p_{i,j} \leq 3\mu$.

$$\therefore \quad x_{i+1} = \phi_{p_i} x_i \ , \quad \mu/4 \leq p \leq 3\mu.$$

Inductively, we have an orbit $O(x_o)$ which admits $\underline{u}$ as a sequence. Thus $\underline{u}$ is in $\sum_o^p$.

Theorem 5.3. Every expansive fixed point free continuous flow on a compact space, is a factor of the flow obtained by suspending the shift on a subspace of $\sum^p$, for some positive integer p.

Proof. Let (X,ϕ) be expansive, with μ-generator

$$\mathcal{F} = \{\overset{o}{F}_i(S_i,\mu)\}_{i=1}^{p}$$

Let $\sum_o^p$ be as above.

We define $\Theta : \sum_o^p \xrightarrow{\text{onto}} \bigcup_i S_i \subset X$ by

$\Theta(\underline{u}) = \pi_{u_o}^{1}(\underline{x}_{\underline{u}})$, where $\underline{x}_u$ is a point in X admitting the sequence $\underline{u}$, (recalling that π_i^{1} denotes the projection from the flow box F_i onto its associated local section S_i).

Θ is well defined, since $\mathcal{F}$ is a generator.

Θ is continuous by Theorem 4.12.

We now define $g : \sum_o^p \to \underline{R}^+$ by

$$g(\underline{u}) = \tau$$

where $\mu/4 \leq \tau \leq 3\mu$ is unique with $\phi_\tau(\Theta(\underline{u}))$ in S_{u_1}.

g is continuous since for $\underline{u}$ close to $\underline{v}$, $u_o = v_o$, $u_1 = v_1$, and $\Theta(\underline{u})$ is close to $\Theta(\underline{v})$, and the rest follows by local product structure of F_{u_o} and F_{u_1}.

Consider the product space $\sum_0^{\;p} \times I_g$ where for each $\underline{u} \in \sum_0^{\;p}$, the fibre above $\underline{u}$ is the interval $[0,g(\underline{u})]$. Form the identification space $Y = \sum_0^{\;p} \times I_{g/\sim}$, where as usual, $(\underline{u}, g(\underline{u})) \sim (\sigma(\underline{u}),0)$.

Let ψ be the flow on Y obtained by suspending the shift σ on $\sum_0^{\;p}$.

Then if we define $p : Y \to X$ by

$$p(\underline{u},r) = \phi_r(\Theta(\underline{u})),$$

p is a homomorphism of (Y,ψ) onto (X,ϕ).

§6. Expansive Flows on 2-manifolds

It is known that the unit interval and the circle do not support expansive homeomorphisms (see e.g. [Jakobsen and Utz 1]).

In this section we prove the following analogue.

Theorem 6.1. A compact (connected) 2-manifold M does not admit a continuous expansive flow.

Proof. We first note that we need only prove the result for orientable surfaces, since a continuous expansive flow on a non-orientable surface M would lift to an expansive flow on $\bar{M}$, the double cover of M, by Corollary 3.32.

Secondly, if $\chi(M)$, the Euler charactistic of M, is not equal to zero, then a continuous flow on M necessarily has a fixed point (see e.g. [Lima 1]) which will be non-isolated. Hence, by Corollary 1.35, Theorem 6.1 is equivalent to the following

Theorem 6.1(a). The cylinder does not admit an expansive flow.

and Theorem 6.1(b). The torus does not admit an expansive flow.

To prove Theorem 6.1(a) we appeal to the Theorem of Poincaré-Bendixson, which states that for a continuous flow in the plane, the only minimal sets of the flow are fixed points and periodic orbits. (A minimal set of the flow is a closed flow invariant subset in which each orbit is dense).

In particular, consider a fixed point free flow ϕ on a cylinder (= annulus) C. Then for all c in C, the α and ω limit sets of c are periodic orbits, which in turn implies that for each c in C, $O(c)$ is either a periodic orbit or $O(c)$ is (weakly) both positively and negatively asymptotic to closed orbits.

We shall show that this means ϕ cannot be expansive via Lemma 6.2.

Let (X,ϕ) be a compact continuous flow and suppose (X,ϕ) has an uncountable number of distinct orbits.
Further we suppose that if x is in X and O(x) is not periodic, then O(x) is weakly positively asymptotic to a periodic orbit and weakly negatively asymptotic to a periodic orbit.

Then (X,ϕ) is not expansive.

Proof. Suppose the contrary. Then there are only a countable number of periodic orbits, say $\{O_1, O_2 \ldots\}$. Let $B(O_m, O_n)$ denote the set of orbits which are weakly negatively asymptotic to O_m and weakly positively asymptotic to O_n. At least one of those sets, $B(O_{m_o}, O_{n_o})$ say, is uncountable.

If $O_{m_o} = O_{n_o}$, select z_o in O_{m_o} and construct $S(z_o)$, $\bar{U} \subset S(z_o)$, and the time change flow ψ of ϕ described in §4.3 (Lemma 4.31).

If $O_{m_o} \neq O_{n_o}$, let A and B disjoint ε-neighbourhoods of O_{M_o}, O_{N_o} respectively, for some $\varepsilon > 0$, and select w_o in O_{m_o}, z_o in O_{n_o}. Then, as in Lemma 4.31, define $\bar{U} \subset S(w_o)$, $\bar{V} \subset S(z_o)$, ensuring that if u is in $\bar{U}$, $\phi_t u$ is in A for $0 \leq t \leq t_u$, where t_u is the time of first return of u to $S(w_o)$. Similarly, for v in $\bar{V}$, $\phi_t(v)$ should be in B for $0 \leq t \leq t_v$. Now construct functions λ_1 on $\bigcup_{u \text{ in } \bar{U}} \{\phi_t u : 0 \leq t \leq t_u\}$ and λ_2 on $\bigcup_{v \text{ in } \bar{V}} \{\phi_t v : 0 \leq t \leq t_v\}$, as in Lemma 4.31.

The domains of λ_1 and λ_2 are disjoint closed subspaces of X, and we construct $\lambda : X \to \underline{R}^+$ such that $\lambda|\text{domain }(\lambda_1) = \lambda_1$, and

$$\lambda|\text{domain }(\lambda_2) = \lambda_2 \quad .$$

Let ψ be the time change flow of ϕ corresponding to λ.

Now (X,ϕ) is expansive, hence there exists $\delta > 0$ such that if x.y are in X and are on different orbits, then for all α and β in hom($\underline{R}$),

$$d(\phi_{\alpha(t)}(x),\phi_{\beta(t)}(y)) > \delta \text{, for some t in } \underline{R}.$$

We now select a local section S_o containing an uncountable set Z, such that, for each z in Z, O(z) is in $B(O_{m_o}, O_{n_o})$, and $z_1 \neq z_2$ implies that $O(z_1) \neq O(z_2)$.

For each z in Z, there exist $s_z < 0$, $t_z > 0$, in $\underline{R}$ such that, $\psi_{s_z}(z)$ is in $\bar{U}$, $\psi_{t_z}(z)$ is in $\bar{V}$,

and for $t \leq s_z$, $d(\psi_{t+s_z}(z), (\psi_t(w_o)) < \delta/2$

and for $t \geq t_z$, $d(\psi_{t+t_z}(z), \psi_t(z_o)) < \delta/2$.

Let $C_{M,N} = \{z \text{ in } Z \text{ with } \begin{matrix} M - 1 \leq s_z \leq M \\ N - 1 \leq t_z \leq N \end{matrix}\}$

where M is a non-positive integer, N a positive integer.

Now $Z = \bigcup_{M,N} C_{M,N}$

$\therefore$ one of the $C_{M,N}$, say C_{M_o, N_o}, is infinite.

Let $T = \max \{|M - 1|, N\}$ Take z_1 and z_2 in C_{M_o, N_o}.

There exist $\zeta > 0$ and $\delta_1 > 0$ such that $[d(z_1, z_2) < \delta_1$ and $|s - t| < \zeta]$ implies that $d(\psi_t(z_1), \psi_s(z_2)) < \delta$ for $|t|, |s| \leq T$. Moreover, there exists $\delta_2 > 0$ such that $d(z_1, z_2) < \delta_2$ implies that $|s_{z_1} - s_{z_2}|$, and $|t_{z_1} - t_{z_2}|$, are $< \zeta$.

By compactness of S_o, select z_1 and z_2 with $z_1 \neq z_2$ in C_{M_o, N_o} and with $d(z_1, z_2) < \gamma = \min \{\delta_1, \delta_2\}$.

We now construct γ in hom($\underline{R}$) as follows:

for $0 \leq t \leq t_{z_1}$, $\gamma(t) = t.\dfrac{t_{z_2}}{t_{z_1}}$;

for $s_{z_1} \leqslant t \leqslant 0$, $\gamma(t) = \dfrac{t.s_{z_2}}{s_{z_1}}$;

for $t > t_{z_1}$, i.e. for $t = s + t_{z_1}$, $s > 0$,

$$\gamma(t) = s + t_{z_2} \quad ;$$

for $t < s_{z_1}$, i.e. for $t = s + s_{z_1}$, $s < 0$,

$$\gamma(t) = s + s_{z_2} \quad .$$

Then, by the various inequalities above,

$$d(\psi_t\, z_1, \psi_{\gamma(t)}\, z_2) < \delta \text{ for all } t \text{ in } \underline{R}.$$

Thus, we have α and β in hom($\underline{R}$) with

$$d(\phi_{\alpha(t)}(z_1), \phi_{\beta(t)}(z_2)) < \delta \text{ for all } t \text{ in } R,$$

contradicting expansiveness of ϕ.

We now consider a fixed point free flow ϕ on the 2-torus T^2. We shall prove Theorem 6.1(b) by considering 2-cases.

(1) There exists a closed ϕ-orbit.

(2) Otherwise.

Case (1). If O_1 is a closed orbit, then O_1 is not null homotopic, otherwise (T^2,ϕ) would have a fixed point, (since it would bound a flow invariant 2-disc). Therefore O_1 bounds a cylinder. Cut T^2 along this cylinder and this gives us a flow on the cylinder which by Theorem 6.1(a) cannot be expansive.

Case (2). We construct a global section for ϕ.

Since T^2 is a 2-manifold, a local section is an arc (i.e. homeomorphic to I, the unit interval) [Whitney 1]. We adopt the following notation.

Let the local section through x_o in T^2 be denoted by $[y_o, z_o]$. Let $\gamma : [-1,1] \to [y_o, z_o]$ be a homeomorphism with $\gamma(0) = x_o$, $\gamma(-1) = y_o$, $\gamma(1) = z_o$.

Since T^2 is compact, and ϕ is fixed point free, there exist μ and $\delta > 0$ such that, for all x in T^2, there exists a local section S_x through x, with associated flow box $F_x = S_x \times [-\mu,\mu]$, such that $F_x' = S_x \times [-{}^{\mu}/2, {}^{\mu}/2]$ contains a ball neighbourhood of x of radius δ.

Choose x_1 in T^2. Construct S_{x_1} through x_1. Then $S_{x_1} = [y_1, z_1] = \gamma_1[-1,1]$ say.

Through z_1 construct another local section S_{z_1}:
$S_{z_1} = [y_2, z_2] = \gamma_2[-1,1]$ say.

Since S_{x_1} and S_{z_1} are local sections, there exists $\varepsilon > 0$ such that either $\gamma_2([-\varepsilon,0]) \cap F_{x_1} = \emptyset$, or $\gamma_2([0,\varepsilon]) \cap F_{x_1} = \emptyset$, but not both. By interchanging the nomenclature of y_2 and z_2 if necessary, we may suppose that $\gamma_2([0,\varepsilon]) \cap F_{x_1} = \emptyset$.

Let $C_2 = [y_1, z_1] \cup \gamma_2([0,1])$,

$$D_1 = F_{x_1}' = S_{x_1} \times [-\mu/2, \mu/2].$$

We have two (mutually exclusive) cases:

(1) There exists t in $(0,1]$ such that $\gamma_2(t)$ is in D_1.

(2) For all t in $(0,1]$, $\gamma_2(t) \cap D_1 = \emptyset$.

If case (2) occurs, we repeat the construction again; viz. construct another section $S_{z_2} = [y_3, z_3]$ $= \gamma_3[-1,1]$ say. through z_2 and consider

$$C_3 = C_2 \cup \gamma_3\,[0,1].$$

$$D_2 = C_2 \times [-\mu/2, \mu/2].$$

and again we have two possibilities.

(1) There exists t in $(0, 1]$ with $\gamma_3(t)$ in D_2.

(2) For all t in $(0,1]$, $\gamma_3(t) \cap D_2 = \emptyset$.

We continue this process until Case (1) occurs. (We remark that case (1) occurs after a finite number of steps, by compactness of T^2).

Consider the first $N > 0$ such that there exists t in $(0,1]$ with $\gamma_{N+1}(t)$ in D_N.

Let $t_o = g\ell b\{t \in (0,1| \text{ with } \gamma_{N+1}(t) \in D_N\}$.
Let $\gamma_{N+1}(t_o) = w$ say. Then w is in D_N.

In particular, w is in $[z_i, z_{i+1}] \times [-{}^{\mu}/2, {}^{\mu}/2]$ for some integer i (if we put z_o equal to y_1).

Therefore $w = \phi_\tau \; w_o$, for w_o in $[z_i, z_{i+1}]$, and $|\tau| \leqslant {}^{\mu}/2$.

If γ is a homeomorphism from $[0,1]$ onto $[w_o, z_{i+1}]$, then the arc $\{\phi_{(1-t)\tau}\gamma(t) : 0 \leqslant t \leqslant 1\} = A_o$ say, is a local section.

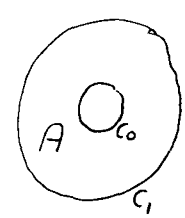
Let C be the closed curve

$$(C_N \setminus C_{i+1}) \cup \gamma_{N+1}([0,t_o]) \cup A_o .$$

By construction C is a local section.

We claim that C is in fact a global section.
We have to show that for each x in T^2, the orbit of x meets C in positive time.

Cut T^2 along C. C is not null homotopic, for if so C would bound a disc, and then ϕ would have a fixed point. Therefore C bounds a cylinder (= annulus) A.



We construct a flow on the plane by extending the orbits of ϕ inwards from C_o, and outwards from C_1, Without loss of generality, we may suppose that points on C_o are flowing into A for positive time (towards C_1), and that points on C_1 are flowing out of A for positive time.

Consider x in $A \setminus C_1$. If x does not reach C_1 in positive time, then its ω-limit set is contained in A, and hence is a closed orbit by the Theorem of Poincaré-Bendixson. This would imply that the original flow ϕ on T^2 had a closed orbit, which would be a contradiction.

Thus every point in $A \setminus C_1$ meets C_1 in positive time, and considering the original flow ϕ on T^2 , (T^2,ϕ) admits a circle (C) as global section. But the circle does not admit an expansive homeomorphism. Hence (T^2,ϕ) is not expansive.

This completes the proof of Theorem 6.1(b).

Remark. We have incidentally shown that if ϕ is a fixed point free flow on T^2 , then either there exists a closed orbit, in which case the only minimal sets of (T^2,ϕ) are closed orbits, or (T^2,ϕ) admits S^1 as a global section, in which case a minimal set of (T^2,ϕ) is the suspension of a minimal set for a homeomorphism on S^1.

APPENDIX

Anosov Flows are Expansive

We apologise for not having found a nice functional analytic proof of this result - the difficulty, as always, seems to arise from having to work "modulu'o the orbits."

The material in this section is extracted from [Anosov 1].

§A.1. Anosov Flows. Let W^m be a compact Riemannian m-manifold, $\dot{w} = f(w)$ a differential equation on W resulting in an Anosov flow ϕ on W^m.

We denote the induced flow on the tangent bundle R of W by $\tilde{\phi}$.

The Anosov condition means that R splits (continuously) into three $\tilde{\phi}$-invariant sub-bundles, say X, Y and Z, where Z is the 1-dimensional bundle along the flow, generated by f, and if the tangent plane at $W = R^m_w = X^k_w \oplus Y^\ell_w \oplus Z^1_w$,

for ξ in X^k_w, ζ in Y^ℓ_w, we have

(a) $|\tilde{\phi}_t\xi| \leq a\,|\xi|\,e^{-ct}$, $t \geq 0$, $|\tilde{\phi}_t\xi| \geq b\,|\xi|\,e^{-ct}$, $t \leq 0$.

(b) $|\tilde{\phi}_t\zeta| \leq a\,|\zeta|\,e^{ct}$, $t \leq 0$, $|\tilde{\phi}_t\zeta| \geq b|\zeta|\,e^{ct}$, $t \geq 0$.

k and ℓ denote the dimensions of X^k_w, Y^ℓ_w respectively, for all w, and the constants a,b and c are positive and are the same for all W, ξ, ζ.

We denote $X^k_w \oplus Y^\ell_w$ by V^{m-1}_w, and $X \oplus Y$ by V.

We have inclusions $\chi(w) : X^k_w \hookrightarrow R^m_w$

$$\lambda(w) : Y^\ell_w \hookrightarrow R^m_w$$

and the corresponding bundle inclusions

$$\chi : X \to R, \quad \lambda : Y \to R, \ (\chi,\lambda) : V \to R .$$

We remark that although χ and λ are not smooth inclusions, they are continuous and do have derivatives along the flow

i.e. $D_f\chi(w) = \frac{d}{dt} \left. \chi(\phi_t w) \right|_{t=0}$ and similarly $D_f\lambda(w)$

exist and depend continuously on w.

If ω is in R^m_w, there is a unique representation for ω.

$$\omega = (\chi, \quad \lambda, \quad f) \begin{pmatrix} \xi \\ \zeta \\ \mu \end{pmatrix} \qquad \begin{array}{l} \text{where } \xi \text{ is in } X^k_w \ , \\ \zeta \text{ is in } Y^\ell_w \ , \\ \mu \text{ is in } Z^1_w \ . \end{array} \tag{1.1}$$

We now study the flow $\tilde{\phi}_t$ on R.

From $\tilde{\phi}_t : R^m_w \to R^m_{\phi_t w}$ we pass to

$$(\chi,\lambda, f)^{-1} \tilde{\phi}_t (\chi, \lambda, f) : X^k_w \oplus Y^\ell_w \oplus Z^1_w \to X^k_{\phi_t w} \oplus Y^\ell_{\phi_t w} \oplus Z^1_{\phi_t w} \quad ,$$

and since the bundles X,Y,Z are $\tilde{\phi}$-invariant, this map is represented by a matrix

$$\begin{pmatrix} P(t,w) & 0 & 0 \\ 0 & Q(t,w) & 0 \\ 0 & 0 & I \end{pmatrix} . \tag{1.2}$$

We derive the system of differential equations describing these flows.

The flow $\tilde{\phi}$ in R has the associated equations $\dot{w} = f(w)$ $\mathring{\omega} = f_w(w) \cdot \omega$, where, in the coordinates associated with a chart U_α in W, $\mathring{\omega} = f_w(w) \circ \omega$ is represented by

$$\mathring{\omega}_i^\alpha = \sum_{j=1}^{m} \frac{\partial f_i^\alpha(w)}{\partial w_j^\alpha} \cdot \omega_j^\alpha \quad .$$

We differentiate (1.1).

$$\mathring{\omega} = (D_f\chi,\ D_f\lambda,\ D_f f)\begin{pmatrix}\xi\\ \zeta\\ \mu\end{pmatrix} + (\chi,\ \lambda,\ f)\begin{pmatrix}\mathring{\xi}\\ \mathring{\zeta}\\ \mathring{\mu}\end{pmatrix}$$

From $\mathring{\omega} = f_w(w) \circ \omega$ we have

$$\begin{pmatrix}\xi\\ \mathring{\zeta}\\ \mathring{\xi}\end{pmatrix} = (\chi,\ \lambda,\ f)^{-1}\left[f_w \circ (\chi,\ \lambda,\ f) - (D_f\chi,\ D_f\lambda,\ D_f f)\right]\begin{pmatrix}\xi\\ \zeta\\ \mu\end{pmatrix}$$

Also $f_w \cdot f = D_f f$.

These equations describe the flow (1.2) and hence split onto separate equations for ξ, ζ and μ.

We have $\mathring{\xi} = A(w)\ \xi$. (1.31)

$\mathring{\zeta} = B(w)\ \zeta$. (1.32)

$\mathring{\mu} = 0$. (1.33)

where

$$\begin{pmatrix}A(w) & 0\\ 0 & B(w)\end{pmatrix} = \Big(\chi(w),\ \lambda(w),\ f(w)\Big)^{-1}\Big[f_w(w).\ \big(\chi(w),\ \lambda(w)\big) - D_f(\chi(\omega),\lambda(\omega))\Big]$$

(1.34) .

§A.2. Riemannian Metrics.

Given an arbitrary Riemannian metric on R, Anosov constructs a special metric for R, and associated metrics for X and Y, which have the property that

For all w in W, ξ in X_w^k, ζ in Y_w^ℓ,

$$\frac{d}{dt}|\tilde{\phi}_t\xi| \leq -a\,|\tilde{\phi}_t\xi|, \qquad \text{where a is some}$$

$$\frac{d}{dt}|\tilde{\phi}_t\zeta| \geq a\,|\tilde{\phi}_t\zeta| \qquad \text{constant} > 0 .$$

It follows that with respect to this metric,

$$|P(t,w)| \leq e^{-at},\quad |Q(-t,w)| \leq e^{-at}, \quad \text{for } t \geq 0 .$$

---

§A.3 Local Sections.

Let $h : R \to W^m$ denote the exponential map

$$(w,\omega) \mapsto h(w,\omega)$$

which has the properties

$$h(w,0) = w \tag{3.1}$$

$$h_\omega(w,0) = 1_{R^m_w} \qquad \text{where } h_\omega = dh\cdot j \tag{3.2}$$

$$j \text{ is the inclusion} : R^m_w \to R^{2m}_{(w,0)}$$

$$\omega \mapsto (\omega,0)$$

$$d(h(w,\omega),w) = |\omega|, \text{ for } |\omega| \leq 1 \tag{3.3}$$

(with the appropriate length scale chosen)

The restriction of h to the unit ball in R^m_w is a diffeomorphism of this ball into W^m.

We introduce the following notation.

$$X_\delta^k\ (w) = \{x : x \text{ is in } X_w^k\ ,\ |x| \leq \delta\ \}$$

Also $Y_\delta^\ell(w)$, X_δ, Y_δ, $V_\delta = X_\delta \oplus Y_\delta$ in similar vein.

For a sufficiently small δ, $h \circ (\chi, \lambda) \Big| V_\delta^{m-1}(w)$ is a smooth inclusion for arbitrary w, and its image, $\pi_\delta(w)$ say, is a disc that is nowhere tangent to the vector field f, (and hence is a local section).

We remark that $\pi_\delta(w)$ changes continuously with w, and if w describes an orbit of the flow, $\pi_\delta(w)$ changes smoothly.

We state the following lemma without proof.

Lemma 3.1. Let w_o be in W, and let $V_\delta{}^I$ denote the V_δ bundle (for $\delta > 0$) restricted to the arc $I = \{\phi_t w_o : |t| \leq \delta\}$

There exists $\delta_o > 0$, such that for all positive $\delta < \delta_o$, $h \circ (\chi,\lambda) \Big| V_\delta{}^I : V_\delta{}^I \to W^m$ is a diffeomorphism onto its image, which contains a ${}^{2\delta}/3$ neighbourhood of w_o.

Lemma 3.2. Let w_o be in W and take δ, $0 < \delta \leq \delta_o$.

If w is in $\pi_\delta(w_o)$, then there exists a maximal interval $\Delta \subset \underline{R}$, and a unique homeomorphism $\beta : \Delta \xrightarrow{\text{into}} \underline{R}$ with $\beta(0) = 0$, such that, for all t in Δ, $\phi_t w$ is in $\pi_\delta(\phi_{\beta(t)}\, w_o)$.

(Equivalently, there exists $\alpha : \Delta' \xrightarrow{\text{into}} \underline{R}$, $\alpha(0) = 0$, such that, for all t in $\Delta' = \beta^{-1}(\Delta)$,

$\phi_{\alpha(t)} w$ is in $\pi_\delta(\phi_t w_o)$.).

Proof. We can define β in a (perhaps degenerate) neighbourhood of 0, using Lemma 3.1, and we just extend the domain of definition of β as far as possible.

We shall show that if δ is sufficiently small, then, for $w \neq w_o$, the domain of definition of β cannot be all of $\underline{R}$. This implies that ϕ is expansive.

For suppose ϕ is not expansive, and let $\delta > 0$ be given. If w_o is in W, $\pi_\delta(w_o)$ is a local section through w_o, and we can construct a flow box $F(w_o) = \pi_\delta(w_o) \times [-\mu,\mu]$ say, based on this section.

There exist points w_o and w in W and α' in hom($\underline{R}$) with

$w \neq \phi_t w_o$, $|t| \leq \mu$,

and $\phi_{\alpha'(t)}(w)$ in $F(\phi_t w_o)$ for all t in $\underline{R}$. Then define $\gamma(t)$ as the unique time t' such that $|t' - \alpha'(t)| \leq \mu$ and $\phi_{\alpha(t)} w$ is in $\pi_\delta(\phi_t w_o)$.

Define α in hom($\underline{R}$) by $\alpha(t) = \gamma(t) - \gamma(0)$.

Then $\phi_{\alpha(t)}(\phi_{\gamma(0)}(w))$ is in $\pi_\delta(\phi_t(w_o))$ for all t in $\underline{R}$.

Also $\phi_{\gamma(0)}(w)$ is in $\pi_\delta(w_o)$ and $\phi_{\gamma(0)}(w) \neq w_o$. Thus this homeomorphism α is precisely the homeomorphism α described in Lemma 3.2, and is defined on $\underline{R}$.

We use local sections to go from the dynamical system ϕ on W^m to go to a related (local) dynamical system, say ψ, on $V_\delta = X_\delta \oplus Y_\delta$.

This system describes how different orbits move in relation to each other. First we construct not ψ but another system θ which has the same orbits but a different time scale.

Let w' be in $\pi_\delta(\omega)$. For the map β in Lemma 3.2. we write $\beta(t,w,w')$ for $\beta(t)$, and $\Delta(w,w')$ for the interval Δ.

Now, if w is in W, let (w,x,y) be in $X_\delta \oplus Y_\delta$. Then $h(w, \chi(w)x + \lambda(w).y) = w'$ say in $\pi_\delta(w)$. We define θ_t as follows:

$$\theta_t(w,x,y) = (w_t, x_t, y_t) \text{ where } w_t = \phi_{\beta(t,w,w')}(w)$$

and x_t, y_t are defined by

$$h(w_t, \chi(w_t).\, x_t + \lambda(w_t)y_t) = \phi_t(w')$$

θ_t is the unique (partial) dynamical system on V_δ satisfying

(1) θ_t is defined for t in $\Delta(w, w')$.

(2) In the domain of definition of θ,

$$h \circ (\chi,\lambda) \circ \theta_t = \phi_t \circ h \circ (\chi,\lambda).$$

(3) Under the projection $\pi : X_\delta \oplus Y_\delta \to W^m$ which sends $(w,x,y) \mapsto w$, oriented orbits of θ_t are projected onto oriented orbits of ϕ_t, but velocity is not preserved.

Θ_t is described by a vector field $(\dot{w},\dot{x},\dot{y})$ on V_δ, using properties (2) and (3).

$$\frac{d}{dt}\, h(w,\chi(w).x + \lambda(w).y) = f.h(w,\chi(w).x + \lambda(w).y) \tag{3.31}$$

$\dot{w} = v(w,x,y).f(w)$, where $v(w,x,y)$ is a scalar multiplier. Hence differentiation along Θ_t of functions depending only on w is carried out with the operator $v.D_f$.

The L.H.S. of (3.31) is

$$h_w(w,\chi(w).x + \lambda(w).y)\; v.f(w) +$$
$$h_\omega(w,\chi(w).x + \lambda(w).y)(vD_f\chi(w).x + vD_f\lambda(w).y) +$$
$$h_\omega(w,\chi(w).x + \lambda(w).y)(\chi(w).\dot{x} + \lambda(w).\dot{y}).$$

where

$$h_w = \frac{\partial h}{\partial w}$$

$$h_\omega = \frac{\partial h}{\partial \omega}$$

Therefore
$$M(w,x,y)\begin{pmatrix}\dot{x}\\ \dot{y}\\ v\end{pmatrix} = f\, h\, (w,\chi(w).x + \lambda(w).y), \tag{3.32}$$

where $M = (M_1\; M_2\; M_3)$,

$$M_1 = h_\omega(w,\chi(w).x + \lambda(w).y)\;\chi(w) \quad ,$$

$$M_2 = h_\omega(w,\chi(w).x + \lambda(w).y)\;\lambda(w) \quad ,$$

$$M_3 = h_w(w,\chi(w).x + \lambda(w).y).f(w) + h_\omega(w,\chi(w).x + \lambda(w).y)(D_f\,\chi(w).x + D_f\,\lambda(w).y) .$$

Now $h_\omega(w,\ \chi(w).x + \lambda(w).y) = h_\omega(w,0) + f_1(w,x,y)$

$$= 1_{R_w^m} + f_1(w,x,y),\ \text{where}$$

f_1 is continuous and $f_1(w,0,0) = 0$.

$$h_w(w,\ \chi(w).x + \lambda(w).y) = h_w(w,0) + \frac{\partial}{\partial\omega} h_w(w,\omega)\Big|_{\omega=0} (\chi(x) + \lambda(y))$$

$+ f_2(w,x,y)$, where $f_2(w,0,0) = 0$.

$$= 1_{R_w^m} + \frac{\partial}{\partial w}\ \frac{\partial}{\partial\omega} h(w,\omega)\Big|_{\omega=0} (\chi(x) + \lambda(y)) + f_2$$

$$= 1_{R_w^m} + \frac{\partial}{\partial w}\ 1_{R_w^m} (\chi(x) + \lambda(y)) \qquad + f_2.$$

$$= 1_{R_w^m} + f_2(w,x,y).$$

Therefore $M_1 = \chi(w) + f_3(w,x,y)$, $\qquad f_3(w,0,0) = 0$.

$M_2 = \lambda(w) + f_4(w,x,y)$, $\qquad f_4(w,0,0) = 0$.

$M_3 = f(w) + D_f\chi(w) + D_f\lambda(w) + f_5(w,x,y)$,

where $f_5(w,0,0) = 0$.

We rewrite (3.22).

$$M(w,x,y)\begin{pmatrix} \dot{x} \\ \dot{y} \\ v-1 \end{pmatrix} = f(h(w,\ \chi(w).x + \lambda(w).y)) - M_3(w,x,y).$$

The R.H.S. equals $fh(\quad) - f(w) - D_f\chi(w) - D_f\lambda(w) - f_5(w,x,y)$.

$$\text{Now } f\,h(\quad) = f\,h(w,0) + f_w(h(w,0)).\,h_\omega(w,0).(\chi(w).x + \lambda(w).y)$$

$$+ f_7(w,x,y), \quad f_7(w,0,0) = 0.$$

$$= f(w) + f_w(w)(\chi(w).x + \lambda(w).y) + f_7 .$$

$$\therefore \quad f - M_3 = f_w(w).(\chi(w).x + \lambda(w).y) - D_f\,\chi(w).x - D_f\,\lambda(w).y$$

$$+ f_7 - f_5$$

$$\text{Hence} \begin{pmatrix} \dot{x} \\ \dot{y} \\ v-1 \end{pmatrix} = \Big((\chi(w),\lambda(w),f(w)\Big)^{-1}\Big[f_w(w).(\chi(w),\lambda(w)) - (D_f\chi(w),\ D_f\lambda(w)\Big]\Big($$

$$+ f_8(w,\ x,\ y),\ \text{where } f_8(w,0,0) = 0 .$$

Hence we have the following equations describing Θ.

$$\begin{aligned} \dot{x} &= A(w).\ x + a(w,\ x,\ y) \\ \dot{y} &= B(w).\ y + b(w,\ x,\ y) \\ v &= 1 + c(w,\ x,\ y) \\ \dot{w} &= v.\ f(w) \end{aligned} \tag{3.33}$$

where the functions a,b and c are continuous and $a(w,0,0) = b(w,0,0) = 0$.

We now pass from Θ to the (local) flow ψ on V_δ which is defined to have the following properties:

(1) $\psi_t(w,x,y)$ is defined for t in $\beta(\Delta(w,w'))$, (and for (w,x,y) in V_δ, $w' = h(w,\ \chi(w)x + \lambda(w).y))$.

(2) Under the map $h \circ \langle \chi, \lambda \rangle$, oriented orbits of ψ go into oriented orbits of ϕ, but generally speaking, velocity is not preserved.

(3) $\pi \circ \psi_t = \phi_t \circ \pi$.

We have the following equations describing ψ_t .

$$\begin{aligned} \dot{x} &= A(w).x + a(w, x, y) . \\ \dot{y} &= B(w).y + b(w, x, y) . \\ \dot{w} &= f(w) \quad . \end{aligned} \tag{3.34}$$

The functions a and b are different from those in (3.33) but they are continuous and satisfy

$$a(w,0,0) = b(w,0,0) = 0 .$$

We may write the transformations $\{\psi_t\}$ as

$$\begin{aligned} x &\to P(t,w).x + p(t, w, x, y) \\ y &\to Q(t,w).y + q(t,w, x,y) \\ w &\to \phi_t w \end{aligned} \tag{3.35}$$

We now compare the flows $\tilde{\phi}$ and ψ by finding estimates for p and q.

<u>Lemma 3.3</u>. For all $\varepsilon > 0$, there exists $\delta > 0$, such that if ψ is the partial dynamical system on V_δ given by (3.35), then $|p|$ and $|q|$ are less than ε, for $-1 \leq t \leq 1$.

<u>Proof</u>. We consider t with $0 \leq t \leq 1$. (For $t \in [-1,0]$, the proof is similar).

let $\zeta(t) = \psi_t(y) - Q(t,w).y$.

Then $\qquad \dot{\zeta}(t) = B(\psi_t(w)).\ \zeta(t) + b(\psi_t(w),\ \psi_t(x),\ \psi_t(y))$.

Thus: $\qquad \zeta(t) = Q(t,w) \int_0^t Q^{-1}(\tau,w).\ b(\psi_\tau(\omega),\psi_\tau(x),\psi_\tau(y))\, d\tau$.

$$= \int_0^t Q((t-\tau),\ \psi_\tau(w)).\ \ b(\quad)\, d\tau$$

Thus $|\zeta(t)| \leq t.\ \max\limits_{0 \leq \tau \leq 1} |Q(\tau,w)| \ \sup\limits_{(w,x,y)} \left| b(w,x.y) \right|$

For δ sufficiently small, $\sup\limits_{(w,x.y)} |b(w,x.y)|$ is sufficiently small for $|\zeta(t)|$ to be less than ε, for $0 \leq t \leq 1$.

Similarly for $\xi(t) = \psi_t(x) - P(t,w).x$.

We recall that $|P(t,w)| \leq e^{-at}$ $\qquad$ for $t \geq 0$,

$|Q(-t,w)| \leq e^{-at}$ $\qquad$ where a is real, positive.

Hence $P(1,w)$ contracts x-vectors by a factor greater than 1, and $Q(-1,w)$ contracts y-vectors by a factor > 1.

Therefore, if δ_1 is chosen sufficiently small, the flow ψ_t will also have the property that for (x,y) in V_{δ_1}, ψ_1 contracts x-vectors by a factor greater than 1, and ψ_{-1} contracts y-vectors by a factor greater than 1.

Hence, by choosing δ smaller than δ_1 if necessary, the flow ψ on V_δ will have the property that

$$\left.\begin{array}{l}|\psi_{-1}(x)| \geq k|x| \\ |\psi_{1}(y)| \geq k|y|\end{array}\right\}, \quad \text{where } k > 1 .$$

This means that if (w,x,y) is in V_δ, then there exists an integer p such that

$\psi_p(w,x,y)$ is not in $V_{\delta/2}$ say.

i.e. $|\psi_p(x)| > \delta/2$, or $|\psi_p(y)| > \delta/2$,

and thus

$\phi_{\alpha(p)}\ h\circ(\chi,\lambda)\ (w,x,y)$ is not in the ball neighbourhood of $\phi_p(w)$ of radius $\delta/3$, by the properties of the flow ψ and Lemma 3.1.

As shown above, this implies that ϕ is expansive on W.

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